Discussion Questions and Problems for "Exploring the Topology of the Projective Plane"

(Revision: March 12, 2018[1])

On 24 March 2018 *Math Counts* will discuss "Exploring the Topology of the Projective Plane"[2]. The following questions are based on Stephen Barr's fun book "Experiments in Topology". Your attempts to address them will help guide our discussion. If the book or these questions are unclear, please ask for help on the event page[2].

1. Inspired by the book, the questions that follow, or your own initiative, what additional experiments did you attempt? What did you learn from these supplemental experiments?

2. According to chapter 1 of Barr's book, what is the rule for homeomorphically distorting one surface into another? What caveats does Barr mention for using his rule? How does Barr's definition compare to Wikipedia's definition of *homeomorphism*?

3. What is the problem with the point x in the picture of the self-intersection for a Klein bottle or a projective plane depicted in Figure 1 on page 79? Why is the point x "a little dubious"?

4. How is a projective plane with a hole in it deformable to a Möbius strip? Is the projective plane with a hole in it, therefore, homeomorphic to the Möbius strip?

5. How is a sphere with a hole in it deformable to a plane? Is a sphere with a hole in it, therefore, homeomorphic to a plane?

6. On page 79, Barr briefly describes a *cross-cap*. What is a cross-cap? How would you describe it? How would you describe it to a child?

7. On pages 79–82, Barr describes the deformation of a Möbius band to the connectivity diagram of a projective plane. Does this mean that the projective plane is homeomorphic to a Möbius band? Or is it the projective plane with a hole in it that is homeomorphic to a Möbius band? What is the point of this carefully described deformation? What are its implications?

8. The text suggests that the figure on the left of Figure 1 on page 79 represents a projective plane. Does it? Even if you interpret the "fake intersection" as it abstractly "should be"? How can you explain this figure as a projective plane? How would cutting the corner y off the figure give a cross-cap?

9. Why does Barr describe the end result of the deformation on pages 79–82 as "a sphere with one cross-cap"? What does that mean? Why is it "a somewhat distorted projective plane"? What is distorted about it?

10. On pages 82–85, Barr describes the Martin Gardner model of a projective plane (Figures 8–11). Do you understand the model? How does it work? How should your imagination interpret the model to "see" the inherent self-intersection that is required?

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[1]Note: Please let me know of any difficulties. There may be a revised version correcting issues if any are found. Compiled by CJ Fearnley. `http://blog.CJFearnley.com`.

[2]`https://www.meetup.com/MathCounts/events/248142433/`

11. On page 83, Barr asks "why does a Möbius strip with only 1 twist give, when cut down the middle on its axis, a loop with 4 twists?" Does he mean 2 half-twists or 4 (does the explanation of a "twist" on page 82 clarify)? That seems to agree with the descriptions on pages 48 and 76, but the definition of a twist has changed? What is going on?

12. On page 85 in referring to the Gardner model in Figure 11, Barr asks "but the edge BA' is half in front and half behind the edge AB': is this allowable?" Can the required joints be made? If so, does it matter that the model is cut and distorted so each half goes separately to its join? Does that entail a discontinuity? Is it OK?

13. In Figures 12, 13, and 14 on pages 85–86, a Möbius band with a cut along half its length is experimented with. What is the purpose of the experiment? Can you duplicate the result of only a 2 twist cylinder? What is your interpretation of this result?

14. In Figure 15 on page 86 another experiment with slits is attempted. What are your results in duplicating the experiment? Do you also find a cylinder with no twists? What are the implications of these experiments?

15. In Figure 20 on page 88 and the associated text, a variation of the Gardner model is described. What is different about this model? Does it fix the problem with the cut flaps going to opposite sides? What are the results of the experiment? What does it mean?

16. In Figure 23 on page 90 and the associated text, a flat disk model of the projective plane is described. How can we interpret this as a projective plane? What are the implications of the dissection along aa'?

17. In figures 26 and 27 on page 93 and in the associated text, a set of experiments exploring the effects of cuts in cruciform models of the projective plane is described. What are the results?

18. In figure 28 on page 94 and the associated text, a "boned" version of the Gardner model is described. How should we interpret the cuts in the Gardner model when they are removed completely? What is the result of this experiment? What does it mean? What does it imply?

19. In Figure 29 on page 95 and the associated text, a 2 piece model of a Möbius strip is described. What is the effect of axial cuts along the strip? Can you now state the rule relating bisection of these surfaces to the number of twists in the result?

20. In several of the experiments discussed in Chapter 6 on "The Projective Plane" subtleties associated with connectivity when cutting and re-joining models are explored. What are these subtleties? What is the topological point of exploring them in our paper models? What caveats must we keep in mind when interpreting a model with cuts in it?

21. The point of the latter part of Chapter 6 is to determine that the projective plane and the Möbius strip are symmetric. What does it mean for a topological surface to be symmetrical? Are you convinced they are symmetrical? How could you explain this symmetry with (paper) models? Could you explain it clearly enough to convince a child that both the Möbius strip and the projective plane are symmetrical?

22. How are the topological and geometrical projective planes related? Can you see both in the models at `http://blog.cjfearnley.com/2012/07/24/models-of-projective-geometry`?