Discussion Questions for "The Symmetry of the Projective Plane (and the curious property of twist)"

(Revision: May 17, 2018[1])

On 26 May 2018 *Math Counts* will discuss "The Symmetry of the Projective Plane (and the curious property of twist)"[2]. The following questions are based on Stephen Barr's fun book "Experiments in Topology". Your attempts to address them will help guide our discussion. If the book or these questions are unclear, please ask for help on the event page[2].

1. On pages 82–85, Barr describes the Martin Gardner model of a projective plane (Figures 8–11). Do you understand the model? How does it work?

2. On page 85 in referring to the Gardner model in Figure 11, Barr asks "but the edge BA' is half in front and half behind the edge AB': is this allowable?" Does that entail a discontinuity? Is there some disconnectedness? Is it OK?

3. On page 83, Barr asks "why does a Möbius strip with only 1 twist give, when cut down the middle on its axis, a loop with 4 twists?" Does he mean 2 half-twists or 4 (does the explanation of a "twist" on page 82 clarify)? That seems to agree with the descriptions on pages 48 and 76, but the definition of a twist has changed? How can we count the number of twists in a topological surface? What is going on?

4. In Figures 12 and 15 on pages 85–86, two oblong Gardner models where we do not glue one pair of edges together are explored. Because only one pair of opposite sides are glued, it is a Möbius strip. Can you duplicate the results of the experiments? Did you get both a two twist and a no twist cylinder? What is your interpretation of these results? What are the implications? Is it OK to cut slits in our models so long as we re-attach them with the correct connectivity as suggested by the definition of homeomorphism given on pages 4–5 in Chapter 1 of the book? Why are the number of twists different in the two cases? Is twist a topological property? Is it part of homeomorphism or geometry?

5. In Figure 19 on page 87 and in Figure 20 on page 88 and the associated text, two variations of the Gardner model are described. Do these models fix the problem with the cut flaps going to opposite sides (see question #2)? Do you also get a loop with two twists when you cut these models as in Figures 13 and 15 on pages 85 and 86 (see question #4)? Why is that? What does it mean?

6. In the circular form of Gardner's model, Figure 23 on page 90 and the associated text, a flat disk model of the projective plane is described. How can we interpret this as a projective plane? What are the implications of the dissection along aa'? Can every projective plane be so dissected? Why? Why not? Does that suggest that the projective plane is asymmetrical?

7. In Figures 26 and 27 on page 93 and in the associated text, a set of experiments exploring the effects of axial cuts in cruciform models of the projective plane is described. What are the results and implications?

[1]Note: Please let me know of any difficulties. There may be a revised version correcting issues if any are found. Compiled by CJ Fearnley. `http://blog.CJFearnley.com`.

[2]`https://www.meetup.com/MathCounts/events/250256203/`

8. In Figure 28 on page 94 and the associated text, a "boned" version of the Gardner model is described. How should we interpret the cuts in the Gardner model when they are removed completely? What are the results and implications of this experiment?

9. In Figure 29 on page 95 and the associated text, a two-piece model of a Möbius strip is described. What is the effect of axial cuts along the strip? Can you now state the rule relating axial bisections of these surfaces to the number of twists in the result?

10. Pages 96–102 describe the experiment of giving paper strips various numbers of twists, then glueing their ends together, then cutting along their axes, and then counting the number of twists in the result. Can you duplicate the experiment and its results? Why does "increasing an odd number of twists by one fail to increase the final number after cutting"? Why does increasing an even number of twists by one add "4 new twists" after cutting? Why do all the strips with $2n + 1$ twists for n a counting number, give, after cutting, a loop with a knot in it? How does this analysis apply to axially cutting the Möbius band as explored in question #3, the oblong Gardner models in #4, the flat Möbius stip in #9, the cruciform models in #7, and the boned version of the Gardner model in #8? How could you explain this twisting rule to a child so they might understand it?

11. On pages 102–105, the circular form of the Gardner model (see question #6) is re-examined. Do you understand the cut in that model that gives the symmetrical Möbius strip of Figure 13 on page 85 (see question #4)? Do you understand the implications of cutting through the center of this model (point C in Figure 23 on page 90)? Do you see how to cut a right- and left-handed Möbius strip out of this model by including more in the cross-cap piece than what would be given by a straight cut through the center? Do you see how to cut the cruciform models of the projective plane out of this model (see Figure 26 and 27 on page 93 and question #7)? Do these considerations "prove" that the projective plane is really symmetrical?

12. On pages 101–102, and again on 105–106, the boned Gardner model is re-examined (see Figure 28 on page 94 and question #8). In Figure 44 on page 106, several "bonings" of other variations of the Gardner model are considered. How do these models support the case that the projective plane is symmetrical?

13. The conclusion of Chapter 6 is that the projective plane and the Möbius strip are symmetrical. What does it mean for a topological surface to be symmetrical? Are you convinced? How could you explain this symmetry clearly enough to convince a child that both the Möbius strip and the projective plane are symmetrical?

14. Does the sequence of experiments with various variations of Martin Gardner's model of the projective plane reveal how imperfect and "lowly" (as Barr calls them on page 106) paper models can help one identify important abstract topological properties? Can geometrical properties such as twist and embeddings in 3-space help the experimental topologist explore and more deeply understand topological surfaces? Does this effort demonstrate that mathematics could be (or maybe even is) an experimental science? Is abstract mathematics simply putting in order the results of a large number of examples?

15. How are the topological and geometrical projective planes related? Can you see the topological perspective in each of the geometrical models described at `http://blog.cjfearnley.com/2012/07/24/models-of-projective-geometry`?