Discussion Questions for "Betti Numbers and the Symmetry of the Projective Plane"

(Revision: July 18, 2018[1])

On 28 July 2018 *Math Counts* will discuss "Betti Numbers and the Symmetry of the Projective Plane"[2]. The following questions are based on Stephen Barr's fun book "Experiments in Topology". Your attempts to address them will help guide our discussion. If the book or these questions are unclear, please ask for help on the event page[2].

1. On pages 123, 124, and 126, Barr provides three definitions for Betti numbers. The first based on the number of edges that need to be removed from a network or graph to produce a connected tree. The second based on cross-cuts and the third based on loop-cuts. Are all three definitions topologically equivalent? Why? Why not?

2. How (in what ways) do the three definitions of Betti numbers characterize the connectivity of a topological surface?

3. What is the Betti number for a disk, an annulus, a torus, a Möbius strip, a Klein bottle, a sphere, and a projective plane? Do the different results reasonably characterize the connectivity of each surface? How? Why?

4. Pages 96–102 describe the experiment of giving paper strips various numbers of twists, then glueing their ends together, then cutting along their axes, and then counting the number of twists in the result. Can you duplicate the experiment and its results? Why does "increasing an odd number of twists by one fail to increase the final number after cutting"? Why does increasing an even number of twists by one add "4 new twists" after cutting? Why do all the strips with $2n + 1$ twists for n a counting number, give, after cutting, a loop with a knot in it? How could you explain this twisting rule to a child so they might understand it?

5. How does this analysis apply to axially cutting the Möbius band as explored on pages 83, 48 and 76, the oblong Gardner models in Figures 12, 13, and 15 on pages 85–86, the flat Möbius strip (Figure 29 on page 95), the cruciform models (Figures 26 and 27 on page 93), and the boned version of the Gardner model (Figure 28 on page 94)?

6. On pages 102–105, the circular form of the Gardner model (see Figure 23 on page 90) is re-examined. Do you understand the cut in that model that gives the symmetrical Möbius strip (see Figures 13 and 14 on pages 85–86)? Do you understand the implications of cutting through the center of this model (point C in Figure 23 on page 90)? Do you see how to cut a right- and left-handed Möbius strip out of this model by including more in the cross-cap piece than what would be given by a straight cut through the center? Do you see how to cut the cruciform models of the projective plane out of this model (see Figures 26 and 27 on page 93)? Do these considerations "prove" that the projective plane is really symmetrical?

7. On pages 101–102, and again on 105–106, the boned Gardner model is re-examined (see Figure 28 on page 94). In Figure 44 on page 106, several "bonings" of other variations of the Gardner model are considered. How do these models support the case that the projective plane is symmetrical?

[1]Note: Please let me know of any difficulties. There may be a revised version correcting issues if any are found. Compiled by CJ Fearnley. `http://blog.CJFearnley.com`.

[2]`https://www.meetup.com/MathCounts/events/252093681/`

8. The conclusion of Chapter 6 is that the projective plane and the Möbius strip are symmetrical. What does it mean for a topological surface to be symmetrical? Are you convinced? How could you explain this symmetry clearly enough to convince a child that both the Möbius strip and the projective plane are symmetrical?

9. Does the sequence of experiments with various variations of Martin Gardner's model of the projective plane reveal how imperfect and "lowly" (as Barr calls them on page 106) paper models can help one identify important abstract topological properties? Can geometrical properties such as twist and embeddings in 3-space help the experimental topologist explore and more deeply understand topological surfaces? Does this effort demonstrate that mathematics could be (or maybe even is) an experimental science? Is abstract mathematics simply putting in order the results of a large number of examples?

10. How are the topological and geometrical projective planes related? Can you see the topological perspective in each of the geometrical models described at `http://blog.cjfearnley.com/2012/07/24/models-of-projective-geometry`?