

Explorations to Define a Theory of Foldable Great Circle Origami

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<http://www.CJFearnley.com/folding.great.circles.2008.pdf>

Abstract

In the folklore of R. Buckminster Fuller's Synergetics, the great circle "railroad tracks" transit "energy" inwardly and outwardly through the center of a sphere or omni-directionally around the great circles or to other inter-connected systems. Fuller illustrated these ideas with models built by folding unit circle modules into spherical cones that exactly dovetail to generate great circle tessellations of the sphere. These tangible models show whole planar great circles (not just the surface graph) and the relationships between angle, volume, and the spherical nets.

We observe that not all great circle nets can be partitioned into unit Eulerian circuits (the modules) with exactly 360° of arc and therefore cannot be folded as elegantly as some of Fuller's examples. Then we describe our investigation to mathematically characterize great circle foldability: to define the surprising relationships between local behavior (unit circle modules) and global structure (great circle nets). Our preliminary results suggest that the fundamental theorem of the foldability of great circle nets may be related to Fuller's "equators of spin" defined by (diametrically opposite) topological elements as poles (vertexes, mid-points of edges, and "centers" of faces).

Explorations to Define a Theory of Foldable Great Circle Origami

Outline of the Talk

1 Introduction

- The Synergetics Collaborative
- What is Synergetics?
- Synergetics and Mathematics

2 Fuller's 8 Foldable Models

- Four Models Derived From the Vector Equilibrium
- Three Models Derived From the Icosahedron
- The Rhombic Triacontahedron
- Summary

3 Seeking a Theory of Foldability

- Joint Work with Jeannie Moberly
- Finding More Foldable Systems
- A simple, symmetrical unfoldable tessellation
- A systematic study of foldability
- Conclusion

The Synergetics Collaborative (SNEC)

- The Synergetics Collaborative (SNEC) is a 501(c)(3) non-profit organization dedicated to bringing together a diverse group of people with an interest in Buckminster Fuller's Synergetics in face-to-face workshops, symposia, seminars, pow-wows, and other meetings to educate and support research and understanding of the many facets of Synergetics, its methods and principles.
- Next Event: **Building a Dome on Bucky's Birthday:** July 11–14, 2008 in Oswego, NY
- Report on our August 2006 Workshop on *Applied Synergetics*:
<http://synergeticists.org/snec.meeting.2006.08.html>
- Report on our July 2005 Workshop on *Structure*:
<http://synergeticists.org/snec.meeting.2005.07.html>
- Report on our July 2004 Workshop on *Tensegrity Structures*:
<http://synergeticists.org/snec.meeting.2004.07.html>

Notable Quotable

Dare to be Naïve!

— Buckminster Fuller

Comprehensive Thinking

Synergetics is the system of comprehensive thinking which R. Buckminster Fuller introduced and began to formulate (primarily, in his two volume magnum opus, Synergetics: Explorations in the Geometry of Thinking, 1975, 1979).

A mathematical model of how the world *really* works

Synergetics is an attempt to build a realistic, comprehensive mathematical modeling system to help guide Humanity to better understand our Universe and to achieve omni-success (“to make the world work for 100% of humanity in the shortest possible time, with spontaneous cooperation and without ecological damage or disadvantage of anyone” — R. Buckminster Fuller).

Humanistic and Philosophical

“Nature’s coordinate system is called Synergetics—synergy means behavior of whole systems unpredicted by any part of the system as considered only separately. The eternally regenerative Universe is synergetic. Humans have been included in this cosmic design as local Universe information-gatherers and local problem-solvers in support of the integrity of the eternal, 100-percent-efficient, self-regenerative system of Universe. In support of their cosmic functioning humans were given their minds with which to discover and employ the generalized laws governing all physical and metaphysical, omniinteraccommodative, ceaseless intertransformings of Universe.”

— Buckminster Fuller

Teleology, Problem-Solving, & Design

Synergetics is teleological: it provides a comprehensive, anticipatory design science method and philosophy for understanding Universe and its applications to problem-solving and design in all areas of human endeavor.

Complex and Multi-Faceted

Synergetics is multi-faceted: it involves geometric modeling, scientific discipline in examining the facts of experience from a fresh perspective, exploring their inter-relationships, and the process of thinking. Synergetics endeavors to identify and understand the principles that Nature actually employs in coordinating Universe (both physically and metaphysically).

The Study of Spatial Complexity

“Synergetics, in the broadest terms, is the study of spatial complexity, and as such is an inherently comprehensive discipline. ... Experience with synergetics encourages a new way of approaching and solving problems. Its emphasis on visual and spatial phenomena combined with Fuller’s wholistic approach fosters the kind of lateral thinking which so often leads to creative breakthroughs.”

– Amy Edmondson, A Fuller Explanation: The Synergetic Geometry of R. Buckminster Fuller, 1987

The Importance of Synergetics to Human Survival

“Most simply put, Synergetics is the study of how nature works, of the patterns inherent in nature, the geometry of environmental forces that impact on humanity. In his thousands of lectures, Fuller urged his audiences to study synergetics, saying **‘I am confident that humanity’s survival depends on all of our willingness to comprehend feelingly the way nature works.’**”

Cheryl Lirette Clark, Ph.D.

12° of Freedom, Ph.D. Thesis

http://www.doinglife.com/12FreedomPDFs/lb_AbstractLitReview.pdf

The Exposition of Synergetics: A Discovery Process

“Synergetics is the study of the way nature, physics and the universe works; it is *‘the geometry of thinking.’* All of Fuller’s work is an exposition of Synergetics following the development of his thinking as it evolved through experiments and practical applications in his artifacts. He says, *‘The omnirational coordinate system which I have named synergetics is not an invention, it is purely discovery.’*”

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Alive and Evolving

Synergetics is alive and evolving in the minds and the work of those who have been inspired by Fuller's exposition.

The **Synergetics Collaborative** is a center for development and education in Synergetics.

Building Upon Bucky's Methods

As with any comprehensive system, Synergetics is incomplete. Bucky gave us a unique insight into what, where & how to explore.

The **Synergetics Collaborative** is working to build on Bucky's Synergetics to expand the work so that it might more effectively fulfill his vision.

Resources

- Both volumes of *Synergetics: Explorations in the Geometry of Thinking* are on-line at

<http://www.rwgrayprojects.com/synergetics/synergetics.html>

- Amy Edmondson's book *A Fuller Explanation: The Synergetic Geometry of R. Buckminster Fuller*

<http://www.angelfire.com/mt/marksomers/40.html>

- Reading Synergetics: Some Tips*

<http://www.cjfearnley.com/synergetics.essay.html>

Notable Quotable

“Mathematics is the science of structure and pattern in general.”

Massachusetts Institute of Technology,
Department of Mathematics

Bucky and Mathematics

- Bucky explored mathematics in ways that are unique and exciting (the geodesic dome, tensegrities, etc.)
- Some of Bucky's choices for language and the lack of axiom/definition based proofs have led the mathematics community to think of him as an outsider

Properties of Synergetics Mathematics

- Synergetics is *integrative*: it seeks relationships between systems and events (in contradistinction to analysis which breaks systems into parts)
- Synergetics is *experiential*: Synergetics Dictionary, Vol. 4, p. 91: "The difference between synergetics and conventional mathematics is that it is derived from experience and is always considerate of experience, whereas conventional mathematics is based upon 'axioms' that were imaginatively conceived and inconsiderate of information progressively harvested through microscopes, telescopes and electronic probings into the non-sensorially tunable ranges of the electromagnetic spectrum."

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Developing the Mathematics in Synergetics

- Today's talks will feature work related to Synergetics and mathematics
- I think that a concerted study of Synergetics can lead to new mathematical results because the text is pregnant with mathematical ideas
- For example, my investigation into Fuller's Foldable Great Circle models reveals one thread through some of the exciting mathematics that lurks just behind the scenes in Synergetics . . . just waiting for someone to look into its nooks and crannies ("Cosmic Fishing").

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Bucky Fuller's Foldable Great Circle Models

- In Synergetics, Fuller demonstrated 8 models which can be constructed by folding modules (with a protractor or origami-style) from whole circles without cutting
- Four (4) of them are characterized by the equators of spin traced by sets of rotating axes each of whose endpoints are diametrically opposite pairs of topological elements (vertices, faces, & edges) of the VE or Vector Equilibrium (what mathematicians call the "Cuboctahedron")

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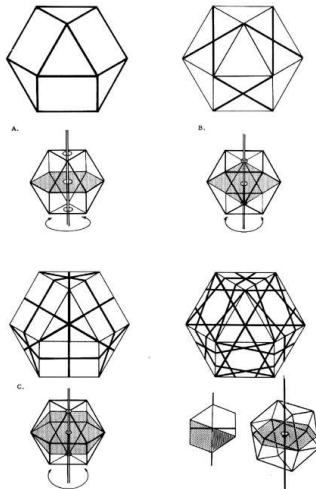


Fig. 450.11A Axes of Rotation of Vector Equilibrium:

- A. Rotation of vector equilibrium on axes through centers of opposite triangular faces defines four equatorial great-circle planes.
- B. Rotation of the vector equilibrium on axes through centers of opposite square faces defines three equatorial great-circle planes.
- C. Rotation of vector equilibrium on axes through opposite vertices defines six equatorial great-circle planes.
- D. Rotation of the vector equilibrium on axes through centers of opposite edges defines twelve equatorial great-circle planes.

Folding the four great circles of the VE

- The four great circle model (formed by axes chosen from the centers of diametrically opposite triangular faces) can be easily folded into “bow-ties” without a compass

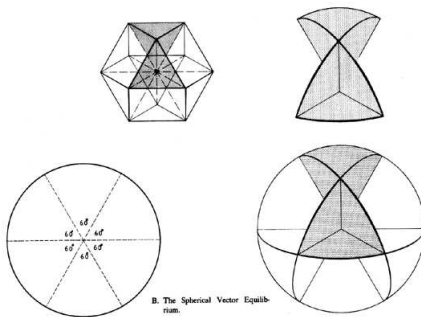


Fig. 455.11 Folding of Great Circles into Spherical Cube or Rhombic Dodecahedron and Vector Equilibrium: Bow-Tie Units:

Folding the three great circles of the VE

- The three great circle model (formed by axes chosen from the centers of diametrically opposite square faces) has each edge doubled

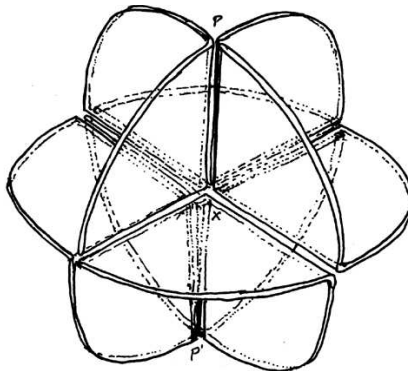
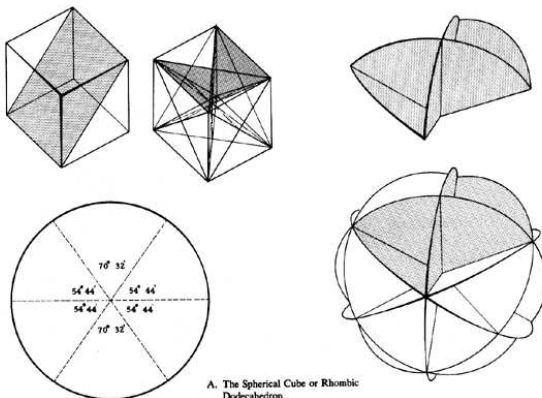


Fig. 835.10 Six Great Circles Folded to Form Octahedron.

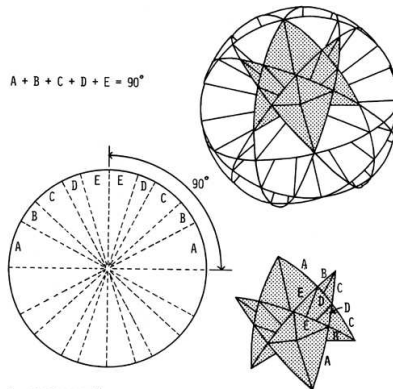
Folding the six great circles of the VE

- This six-great-circle construction defines the positive-negative spherical tetrahedrons within the cube. This also reveals a spherical rhombic dodecahedron. The circles are folded into "bow-tie" units as shown. The shaded rectangles in the upper left indicates the typical plane represented by the six great circles.



Folding the twelve great circles of the VE

$$A + B + C + D + E = 90^\circ$$



$$A = 28.56082521^\circ$$

$$B = 14.45828792^\circ$$

$$C = 19.28632541^\circ$$

$$D = 10.67069527^\circ$$

$$E = 17.02386618^\circ$$

Fig. 450.10 The 12 Great Circles of the Vector Equilibrium Constructed from 12 Folded Units (Shown as Shaded).

Bucky's modules dovetail to form great circle nets

The remarkable thing about these models is how the modules (local foldings of circles) dovetail together to form complete great circle models without cutting the circle into arcs.

- One would expect that although a random folding of a circle would have geodesic arcs, they will not typically dovetail to compose a great circle tessellation of the sphere
- One would expect that a random decomposition of a great circle net on the sphere into disjoint Eulerian circuits (subgraphs that visit each edge exactly once) of exactly 360° of arc would be rare.
- So these models are mathematically **special** in some way

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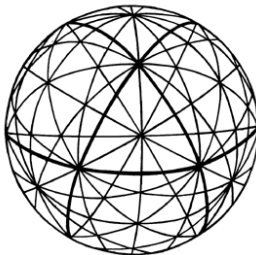
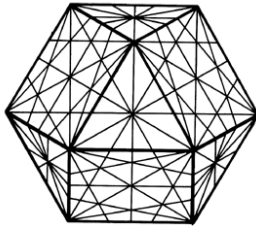
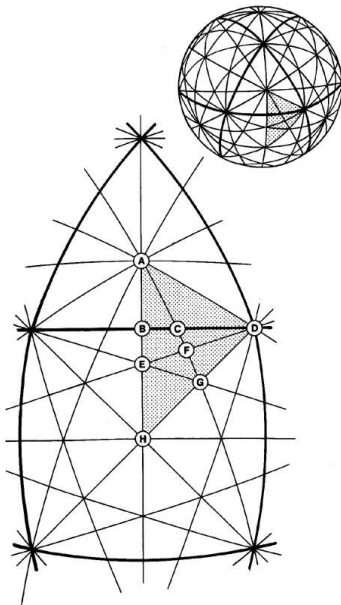


Fig. 450.11B Projection of 25 Great-Circle Planes in Vector Equilibrium Systems: The complete vector equilibrium system of 25 great-circle planes, shown as both a plane faced-figure and as the complete sphere ($3 + 4 + 6 + 12 = 25$). The heavy lines show the edges of the original 14-faced vector equilibrium.



CENTRAL ANGLES		
19.47122063°	AB	19° 28' 16.394"
35.26438968°	AD	35° 15' 51.803"
22.20795430°	AC	22° 12' 27.656"
10.89359465°	BC	10° 53' 36.221"
19.10606535°	CD	19° 06' 25.779"
10.02498786°	BE	10° 01' 29.956"
6.55517091°	CE	6° 33' 11.415"
14.45828792°	EF	14° 27' 29.637"
17.02386618°	FD	17° 01' 25.918"
19.28632541°	EG	19° 17' 10.773"
10.67069527°	FG	10° 40' 14.503"
25.25940182°	FH	25° 14' 21.847"
26.56505118°	HG	26° 32' 54.184"
18.43494882°	GD	18° 26' 5.836"
51.48215410°	DE	51° 28' 55.755"
30.	BD	30° 00' 00"
45.	DE	45° 00' 00"
54.75561031°	AH	54° 44' 8.197"

FACE ANGLES		
30.°	BAC	30° 00' 00.000"
30.	CAD	30° 00' 00.000"
30.	ABC	30° 00' 00.000"
61.87449436°	ACB	61° 52' 28.179"
118.1255057°	ACD	118° 7' 51.821"
35.26438968°	ADC	35° 15' 51.803"
90.	BBC	90° 00' 00.000"
118.1255057°	BCE	118° 7' 51.821"
73.22134512°	BEF	73° 13' 16.842"
80.40593179°	CBE	80° 24' 21.354"
61.87449430°	PCD	61° 52' 28.179"
19.47122063°	CDP	19° 28' 16.394"
40.59406821°	CFD	40° 35' 38.646"
73.22134512°	HCE	73° 13' 16.842"
65.90515745°	EGH	65° 54' 18.567"
45.	EHG	45° 00' 00.000"
90.59406821°	EPG	90° 35' 38.646"
33.55750977°	PHG	33° 33' 26.315"
48.18968511°	FGH	48° 11' 22.846"
80.40593179°	OPD	80° 24' 21.354"
35.26438969°	PDG	35° 15' 51.803"
65.90515745°	PGD	65° 54' 18.567"

Fig. 453.01 Great Circles of Vector Equilibrium Define Lowest Common Multiple Triangle: 1/48th of a Sphere: The shaded triangle is 1/48th of the entire sphere and is the lowest common denominator (in 24 rights and 24 lefts) of the total spherical surface. The 48 LCD triangles defined by the 25 great circles of the vector equilibrium are grouped together in whole increments to define exactly the spherical surface areas, edges, and vertexes of the spherical tetrahedron, spherical cube, spherical octahedron, and spherical rhombic dodecahedron. The heavy lines are the edges of the four great circles of the vector equilibrium. Included here is the spherical trigonometry data for this lowest-common-denominator triangle of 25-great-circle hierarchy of the vector equilibrium.

Folding the great circles of the Icosahedron

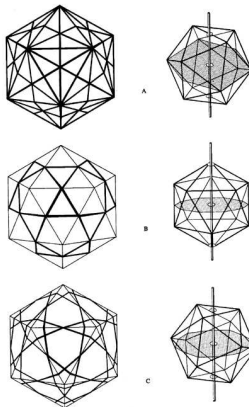


Fig. 457.30A Axes of Rotation of Icosahedron:

- A. The rotation of the icosahedron on axes through midpoints of opposite edges define 15 great-circle planes.
- B. The rotation of the icosahedron on axes through opposite vertices define six equatorial great-circle planes, none of which pass through any vertexes.
- C. The rotation of the icosahedron on axes through the centers of opposite faces define ten equatorial great-circle planes, which do not pass through any vertexes.

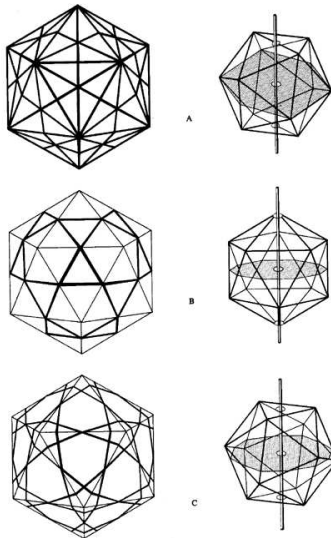


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Three Models Derived From the Icosahedron

Folding the 6 & 15 great circles of the Icosahedron

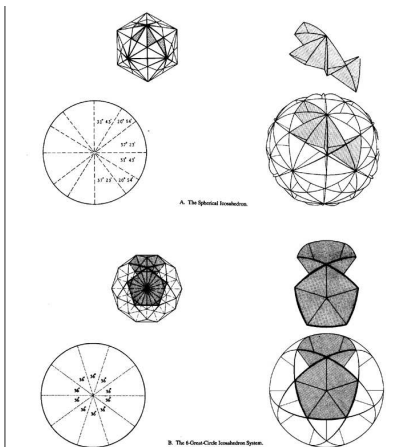


Fig. 458.12 Folding of Great Circles into the Icosahedron System:

- A. The 15 great circles of the icosahedron folded into "multi-bow-ties" consisting of four tetrahedrons each. Four times 15 equals 60, which is 1/2 the number of triangles on the sphere. Sixty additional triangles inadvertently appear, revealing the 120 identical (although right- and left-handed) spherical triangles, which are the maximum number of like units that may be used to subdivide the sphere.
- B. The six great-circle icosahedron system created from six pentagonal "bow-ties."

Folding the 10 great circles of the Icosahedron

Both positive and negative units
can be folded from this circle.
To fold negative unit,
interchange $\ominus$ and $\oplus$

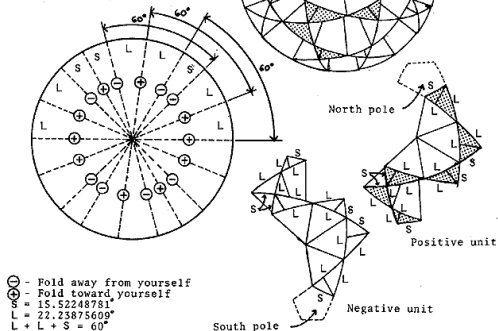


Fig. 455.20 The 10 great circles of the Icosahedron Constructed from 10 folded units (5 positive units + 5 negative units).

The 31 great circles of the Icosahedron

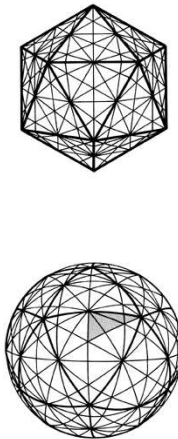
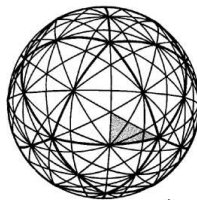


Fig. 457.30B Projection of 31 Great-Circle Planes in Icosahedron System: The complete icosahedron system of 31 great-circle planes shown with the planar icosahedron as well as true circles on a sphere ($6+10+15=31$). The heavy lines show the edges of the original 20-faced icosahedron.



A.

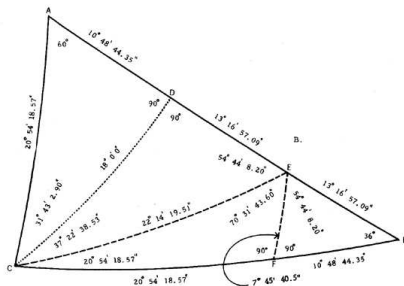


Fig. 901.03 Basic Right Triangle of Geodesic Sphere: Shown here is the basic data for the 31 great circles of the spherical icosahedron, which is the basis for all geodesic dome calculations. The basic right triangle as the lowest common denominator of a sphere's surface includes all the data for the entire sphere. It is precisely 1/120th of the sphere's surface and is shown as shaded on the 31-great-circle- sphere (A). An enlarged view of the same triangle is shown (B) with all of the basic data denoted. There are three different external edges and three different internal edges for a total of six different edges. There are six different internal angles other than 60° or 90°. Note that all data given is spherical data, i.e. edges are given as central angles and face angles are for spherical triangles. No chord factors are shown. Those not already indicated elsewhere are given by the equation $2 \sin(\theta/2)$, where θ is the central angle. Solid lines denote the set of 15 great circles. Dashed lines denote the set of 10 great circles. Dotted lines denote the set of 6 great circles.

Bucky's 30 great circles model

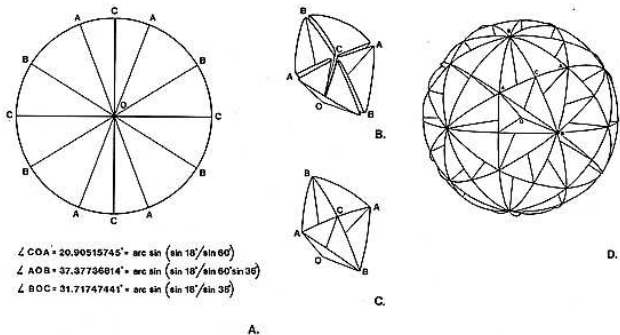


Fig. 986.502 Thirty Great-circle Discs Foldable into Rhombic Triacontahedron System: Each of the four degree quadrants, when folded as indicated at A and B, form separate T Quanta Module tetrahedra. Orientations are indicated by letter on the great-circle assembly at D.

Summary of Fuller's Foldable Models

- Fuller showed 8 Models that form great circle tessellations of the sphere by folding circles into modules
- Fuller's modules essentially partition a great circle graph on a sphere into disjoint Eulerian circuits of exactly 360°
- So is the foldability property rare?
- Since Fuller gave us eight (8) models, maybe foldability is commonplace?
- **My research question:** What great circle tessellations are foldable and why?

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Investigating the Foldability Property

- In 1990 in Binghamton, NY, I was studying Synergetics and built a model of the 25 great circles of the VE with 336 little construction paper arcs.
- In 2004, Jeannie Moberly and I published work (and a proof) that the 25 great circles of the VE are **not** foldable

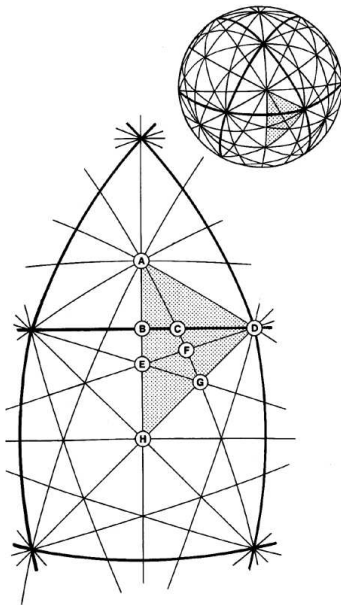
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- In 2004, Jeannie Moberly and I published work (and a proof) that the 25 great circles of the VE are **not** foldable

Notable Quotable

“[Proof is no more than] the testing of the products of our intuition ... Obviously we don't possess, and probably will never possess, any standard of proof that is independent of time, the thing to be proved, or the person or school of thought using it. And under these conditions, the sensible thing to do seems to be to admit that there is no such thing, generally, as absolute truth in mathematics, whatever the public may think.”

— Raymond L. Wilder



CENTRAL ANGLES		
19.47122063°	AB	19° 28' 16.394"
35.26438968	AD	35° 15' 51.803"
22.20795430	AC	22° 12' 27.656"
10.89359465	BC	10° 53' 36.221"
19.10600535	CD	19° 06' 25.779"
10.02498786	BE	10° 01' 29.956"
6.55517091	CE	6° 33' 11.415"
14.45828792	EF	14° 27' 29.637"
17.02386618	FD	17° 01' 25.918"
19.28632541	EG	19° 17' 10.773"
10.67069527	FG	10° 40' 14.503"
25.25940182	FH	25° 14' 21.847"
26.56505118	HG	26° 32' 54.184"
18.43494882	GD	18° 26' 5.836"
51.48215410	DE	51° 28' 55.755"
30.	BD	30° 00' 00"
45.	DE	45° 00' 00"
54.75561031	AH	54° 44' 8.197"

FACE ANGLES		
30.°	BAC	30° 00' 00.000"
30.	CAD	30° 00' 00.000"
30.	ABC	30° 00' 00.000"
61.87449436°	ACB	61° 52' 28.179"
118.1255057	ACD	118° 7' 51.821"
35.26438968	ADC	35° 15' 51.803"
90.	BBC	90° 00' 00.000"
118.1255057	BCE	118° 7' 51.821"
73.22134512	BEF	73° 13' 16.842"
80.40593179	CBE	80° 24' 21.354"
61.87449430	PCD	61° 52' 28.179"
19.47122063	CDP	19° 28' 16.394"
90.59406821	CFB	90° 35' 38.646"
73.22134512	HBC	73° 13' 16.842"
65.90515745	EGH	65° 54' 18.567"
45.	EHG	45° 00' 00.000"
90.59406821	EPG	90° 35' 38.646"
33.55750977	PHG	33° 33' 26.315"
48.18968511	FGH	48° 11' 22.846"
80.40593179	OPD	80° 24' 21.354"
35.26438969	PDG	35° 15' 51.803"
65.90515745	PGD	65° 54' 18.567"

Fig. 453.01 Great Circles of Vector Equilibrium Define Lowest Common Multiple Triangle: 1/48th of a Sphere: The shaded triangle is 1/48th of the entire sphere and is the lowest common denominator (in 24 rights and 24 lefts) of the total spherical surface. The 48 LCD triangles defined by the 25 great circles of the vector equilibrium are grouped together in whole increments to define exactly the spherical surface areas, edges, and vertexes of the spherical tetrahedron, spherical cube, spherical octahedron, and spherical rhombic dodecahedron. The heavy lines are the edges of the four great circles of the vector equilibrium. Included here is the spherical trigonometry data for this lowest-common-denominator triangle of 25-great-circle hierarchy of the vector equilibrium.

The 25 great circles of the VE are NOT foldable

2004: Joint work with Moberly: A model of the 25 great circles of the VE **cannot** be built from 25 equal 360° modules.

Proof: Consider the $\frac{1}{48}$ LCD triangle (or Schwarz triangle which by repeated reflection in their sides covers the sphere a finite number of times) of the 25 great circles of the VE. Each arc interior to the LCD triangle occurs 48 times in the 25 great circles. But each arc on the boundary or edge is shared by two LCD triangles and only occurs 24 times. Since each module must be identical for our construction, we need to associate each of the 25 great circles with the 24 distinct occurrences of each boundary arc. 24 and 25 are incommensurable and so it is impossible.

References:

- Proceedings for the 2004 Bridges: Mathematical Connections in Art, Music, and Science pp. 341–342.
(Also, on-line at <http://www.cjfearnley.com/supercircles.pdf>.)
- Slides for presentations to Bridges and a special Mathematics departmental seminar at Binghamton University: <http://www.cjfearnley.com/supercircles.slides.01.pdf>.

First Steps Toward a Theory of Foldability

- **Combinatorial constraint:** the number of great circles must equal the number of modules or else at least some of the arcs will be doubled (or tripled, etc.). The number of great circles must divide the number of modules unless we allow the modules to be “different”.
- **What is the foldability property?**
 - The spherical tessellation must consist of great circles only
 - The modules must consist of identical units with exactly 360° without “hiding” arc
 - **Option:** Allow doubling, etc. of arc (note 3 GC of the VE)
 - **Option:** Allow handedness differences (note the 10 GC of the icosahedron)
 - **Option:** Allow arbitrary differences in modules (but keep the 360° property for each module)

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Identifying Additional Foldable Tessellations

- The simplest foldable great circle net consists of a single unfolded circle
- N circles folded in half and married along their diameters generates an infinite class of foldable models (provided the arcs are fixed at the “correct” angles to ensure we have great circles and not just a set of half-arcs)
- The VE is **quasiregular** (formed by the intersection of two dual regular polyhedra). So the icosidodecahedron might generate a fourth set of axes for a new great circle tessellation?

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The 30 great circles of the icosidodeca are Foldable

Plate #27

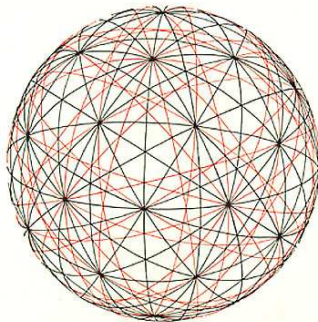
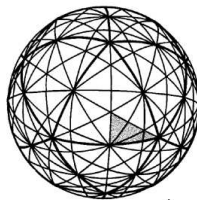


Plate 27 (See Sec. 942.132A)





A.

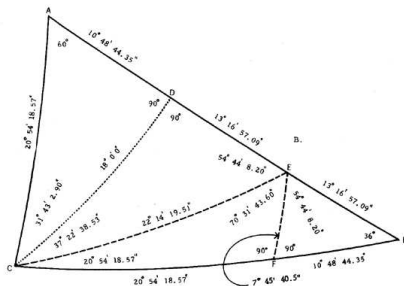


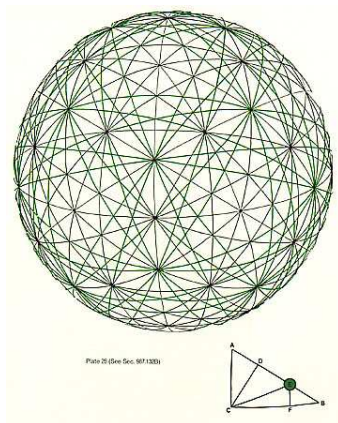
Fig. 901.03 Basic Right Triangle of Geodesic Sphere: Shown here is the basic data for the 31 great circles of the spherical icosahedron, which is the basis for all geodesic dome calculations. The basic right triangle as the lowest common denominator of a sphere's surface includes all the data for the entire sphere. It is precisely 1/120th of the sphere's surface and is shown as shaded on the 31-great-circle- sphere (A). An enlarged view of the same triangle is shown (B) with all of the basic data denoted. There are three different external edges and three different internal edges for a total of six different edges. There are six different internal angles other than 60° or 90°. Note that all data given is spherical data, i.e. edges are given as central angles and face angles are for spherical triangles. No chord factors are shown. Those not already indicated elsewhere are given by the equation $2 \sin(\theta/2)$, where θ is the central angle. Solid lines denote the set of 15 great circles. Dashed lines denote the set of 10 great circles. Dotted lines denote the set of 6 great circles.

Are the secondary great circles of the icosahedron Foldable?

Is the set of equators formed by axes defined by diametrically opposite vertices D , E , and F (each of which is defined by points of intersection in the 31 great circles aggregate) foldable?

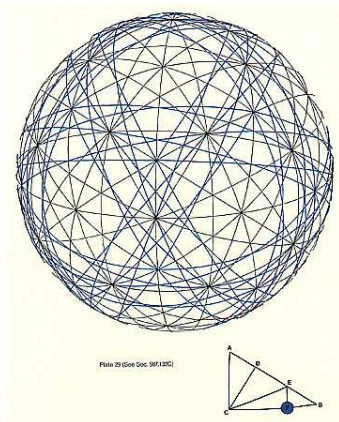
The secondary great circles of the icosahedron are Foldable

Plate #28



The secondary great circles of the icosahedron are Foldable

Plate #29

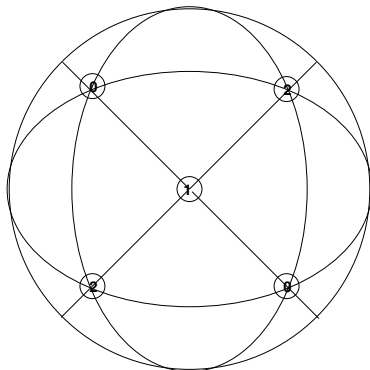


Might the foldability property be related to symmetry?

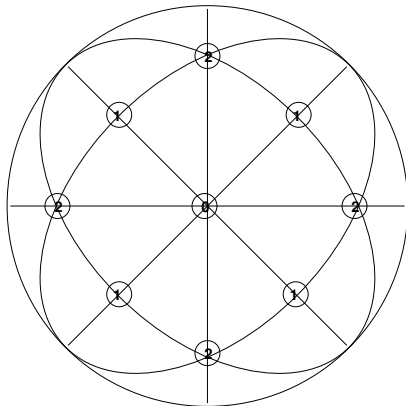
Symmetry: I then asked myself if simple symmetries might generate more foldable tessellations? So I looked in Coxeter's *Regular Polytopes* at the *lines of symmetry* for the regular polyhedra. The lines of symmetry are related to the equatorial polygon dividing a quasiregular polyhedron in half. There are three models to consider.

A simple, symmetrical unfoldable tessellation

Some “Lines of Symmetry” Models are Foldable, Some Are Not



6 Great Circles (lines on symmetry)
Tetrahedron



9 Great Circles (lines of symmetry)
Octahedron / Hexahedron

Do equators of spin generate foldable systems?

Conjecture (from my Abstract): Great circle nets composed from the equators of spin generated by a set of axes defined by one class of topological elements (vertices, faces, or edges) from a “sufficiently symmetrical” polyhedron, can be folded from unit circles into a set of “identical” modules.

Counterexample: Note, there is a one-to-one map (an “injection”) between great circles and the poles of axes, so equators of spin are always extant (if somewhat implicit) in a great circle tessellation. Hence the unfoldability of the 9 lines of symmetry of the octa/cube provides a counterexample to the above conjecture. Thus, just because we have equators of spin from a symmetrical polyhedron, does not mean it is foldable.

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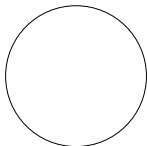
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A systematic study of foldability

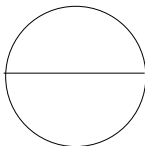
Robert Gray pointed out to me that because of the symmetry of a great circle net, one can choose one of the great circles as an equator and ignore the mirror image on the other side.

Therefore, one can model a great circle net by looking only at a simple circle with lines cutting through it.

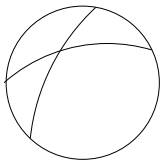
Diagramming Great Circle Nets



One Great Circle

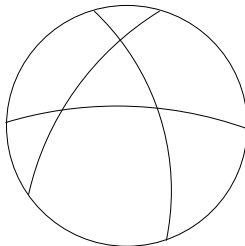


Two Great Circles



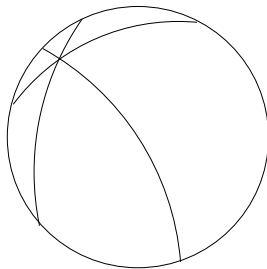
Three Great Circles

(Each interior arc gets 180 degrees
So "slide" the arcs around to get each
possible 3 great circle net.)



Four Great Circles

Are all variations foldable?



Four Great Circles

With fewer intersections
The geometry changes
not foldable???

Concluding Conjecture

Conjecture If the number of great circles equals the number of fundamental regions in a symmetrical tessellation of the sphere, then it can be folded with identical modules.

Note: this is clearly a necessary condition, but is it sufficient?

Open Questions

- How might we precisely define the foldability criterion?
- How might we classify all foldable spherical tessellations?
- Are the secondary great circles of the VE foldable?
- Are all four great circle nets foldable?
- What is the significance of doubling the edges in some models?
- Does there exist a spherical great circle net with the property that several different modules dovetail to produce the whole model? [I think the 15 great circles of the icosahedron might have this property.]

Conclusion

Buckminster Fuller's magnum opus, Synergetics, is pregnant with unexplored mathematical themes which, to date, the mathematics community has largely ignored. My work on foldable great circle tessellations provides an example of how the type of basic mathematics questions implicit in the Synergetics text can result in pioneering mathematical studies that are accessible to even high school students and undergraduates.

Thank You

Thank You!

Any Questions?

<http://www.CJFearnley.com/folding.great.circles.2008.pdf>