This talk describes our attempt to develop *harmonic perspective* as a form of artistic representation.

Next Slide: (A Coxeter Inspired Journey)

We've been studying Coxeter's *Introduction to Geometry* for more than a decade. It is a great work (or maybe we are just not that smart, I'm not sure which).

In Coxeter's book we kept running into a surprising and peculiar property, the harmonic property. Although he introduces it early on, harmonics become a central element in the development of projective geometry. We'll spend some time giving you a little taste for harmonics. But first a quick introduction to projective geometry.

One way to look at projective geometry is as the attempt to eliminate the special case distinctions between interesecting and parallel lines which complicate Euclidean geometry.

Next Slide: (The Duality Principle)

One way to appreciate the benefits of this simplified geometry is with the duality principle: In 2 dimensional projective geometry for any true statement the word "point" and the word "line" can be interchanged (with a suitable change in the language) and the resulting *dual* statement remains true. In 3 dimensions, we swap "point" and "plane" while lines are self-dual.

Next Slide: (Projective Geometry)

Perhaps the most important dual facts in the projective plane are given by its axioms: Between *any* two points there is a unique line. And dually, *any* two distinct lines meet in a unique point. That's the big difference from Affine or Euclidean geometry. It excludes parallel lines but with the benefit of an elegant dual relationship between point and line.

There are many ways to develop projective geometry. My essay "Models of Projective Geometry", published yesterday on my blog, provides another introduction.

Next Slide: (A Reference to Harmonic Perspective)

Harmonic Perspective is our attempt to integrate the basic ideas of perspective with harmonics. Our literature search found a 1913 book by Hatton which got us going.

Next Slide: (Point Perspective)

Perspective is also a dual concept: it is a kind of mapping of points to points, or dually, lines to lines. In the first case, we have point perspective: Two lines are perspective from a point if corresponding points lie on lines through a *center* of perspective.

Next Slide: (Line Perspective)

In the dual situation, Line Perspective: two points are perspective from a line if corresponding lines in each pencil — The lines through a point are a pencil — So two points are perspective from a line if corresponding lines in each pencil meet along the *axis* of perspective.

Next Slide: (Harmonics and Quadrangles)
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Current Slide: (Harmonics and Quadrangles)

Now for a first taste of harmonics. The harmonic property is rare: four randomly choosen points on a line are probably not going to be related harmonically. Harmonics are special, but it's a bit complicated. It was unexpected and puzzling when we first saw it.

A quadrangle (technically, we should say "a complete quadrangle", but as you'll see quadrangles are "hand-in-glove" with harmonics so we'll just call them quadrangles). Quadrangle $PQRS$ has 4 points and 6 sides. Opposite sides (the ones that don't meet at a vertex) meet in three diagonal points B, D, and T. Let's focus in on the side BD of the diagonal triangle. We call the points B & D the diagonal points of a harmonic set.

Draw the line BD. Note B & D are each intersections of pairs of sides of the quadrangle. So four of the six sides of the quadrangle are "taken" by the diagonal points B & D. Where the remaining two sides of the quadrangle (the dashed lines) meet line BD are the conjugate points of the harmonic set. This is the fundamental fourness divided into two pairs: diagonal points B & D with the conjugates A & C. This is the harmonic property expressed by a quadrangle.

The lower half of the diagram shows that the diagonal points and the conjugates can be swapped by using a different quadrangle. Indeed there are many, many quadrangles to show that the pairs B & D and A & C are in a harmonic relationship. So we need to think fluidly about which are diagonal and which are conjugate points: it depends on which quadrangle we look at!

Next Slide: (Here Kitty)

Jeannie:

The artworks are made of hardboard painted with acrylic and oil. I chose multiple intersecting planes to create 3D spaces to emphasize the projective properties and to sidestep the influences of Renaissance and photographic perspective.

In "Here Kitty" quadrangles in various planes intersect in a common harmonic set. Different quadrangles use different diagonal and conjugate points.

CJ:

We will be seeing other ways to look at harmonics shortly. In all cases, there is a complete quadrangle lurking in the background dividing fourness into two pairs: the diagonal points and the conjugate points.

Next Slide: (The Circle of Apollonius)
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Current Slide: (The Circle of Apollonius)

The Circle of Apollonius introduces new ways to see harmonics. First, let's look at the triangle construction of harmonics. Take a triangle such as BDQ and draw its internal and external angle bisectors. (Long pause). Do you remember your Euclidean construction for angle bisectors from high school?

(wait for Jeannie)

The internal and external angle bisectors of triangle BDQ, intersect the opposite side in a harmonic set. A quadrangle showing the harmonic relationship is $PQRS$ where S is an ideal point, a so-called point at infinity (given CQ, we draw its parallels BP and DR). As in the previous diagram B & D are diagonal points with A & C as conjugates.

The locus of points Q whose distance from two fixed points B and D are in a fixed ratio is the circle of Apollonius. Its diameter is between the conjugate points of our harmonic set.

Next Slide: (Harmonic Ratios I)

The fixed ratio in the Circle of Apollonius is $\frac{DQ}{QB}$ = it equals the ratios of lengths in our harmonic set $\frac{DC}{CB} = \frac{DA}{AB}$. This harmonic ratio provides yet another specification for harmonics.

Jeannie:

In a harmonic ratio, one outer segment (such as DC) is to the central segment (CB) as the whole (AD) is to the other outer (AB). It doesn't matter which "outer" you start with, the ratios work out.

So also by taking the first outer to be AB, we have $\frac{AB}{BC} = \frac{AD}{DC}$.

Next Slide: (Harmonic Ratios II)

Jeannie:

For the artist, these *harmonic ratios* provide a means of comparing the sizes of objects in a perspective drawing

I used harmonic ratios to relate all three artworks.

$\frac{\text{person}}{\text{cat}} = \frac{\text{hand}}{\text{needle}}$, $\frac{\text{person}}{\text{window}} = \frac{\text{window}}{\text{cat}}$, $\frac{\text{person}}{\text{scissor}} = \frac{\text{insect}}{\text{hand}}$

Next Slide: (Inside Out)

Jeannie:

Inside Out features an Apollonian circle construction and an inversion through separate conics so the observer observes the observer iteratively.

Next Slide: (Harmonic Perspective)
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Current Slide: (Harmonic Perspective)

This diagram is too complex to fully explain in the limited time. Suffice it to say that we find additional harmonic pencils and harmonic sets and produce part of a cascading grid of harmonics.

This construction also highlights another definition of harmonics. An involution is a transformation of period two. An involution of a line into itself with two fixed points always yields a harmonic set. The two fixed points serve as diagonal points and the points that interchange are the conjugates. There is a dual notion for the involution of a pencil into itself giving a harmonic pencil. Our paper mentions even more characterizations of harmonics. It is a deep, subtle, and multi-dimensional concept!

Jeannie: Here we see how the dual form works with ratios of angles in a harmonic pencil. Which are also very useful and used extensively in this diagram.

Next Slide: (Intentional Cut)

Jeannie (continued):

Which is the basis of my third art piece, "Intentional Cut" where harmonic pencils derived from two lines intersect in a quadrangle whose points act in the involution of conjugate points along the harmonic line on the left which has one distant invariant point.

This first attempt to develop art from harmonic perspective was illuminating, but we think a more satisfying integration of art and projective geometry is possible.

Next Slide: (Unanswered Questions)

CJ:

Is it possible to design art based on a geometrical invariant such as harmonics?

Jeannie:

Can harmonic perspective help us understand the simultaneous perceptions of large and small, near and far objects?

What qualities can harmonic perspective interject into a work of art?

Next Slide: (Conclusion)

CJ:

The many definitions of harmonics provide tools for the artist and the geometer to explore these and other representational questions.

Perspective and harmonics and their interrelationships identify subtleties of spatial experience that speak to the power and importance of projective geometry.

Next Slide: (Thank You)

Thank You.