Discussion Questions and Problems for "Topological Experiments: The Conical Möbius Band & the Klein Bottle"

(Revision: January 19, 2018[1])

On 27 January 2018 *Math Counts* will discuss "Topological Experiments: The Conical Möbius Band & the Klein Bottle"[2]. The following questions and problems are based on Stephen Barr's fun book "Experiments in Topology". Your attempts to address them will help guide our discussion. If the book or these questions are unclear, please ask for help on the event page[2].

1. Inspired by the book, the questions that follow or your own initiative, what additional experiments did you attempt? What did you learn from these supplemental experiments?
2. According to chapter 1 of Barr's book, what is the rule for homeomorphically distorting one surface into another? What caveats does Barr mention for using his rule? How does Barr's definition compare to Wikipedia's definition of *homeomorphism*?
3. What is the property of orientability? Which of the sphere, disk (the 2D surface bounded by a circle), cylinder, torus, and Möbius band are orientable and which are nonorientable?
4. Why are the paper models of the Klein Bottle and Projective Plane in Figures 2 and 3 on p. 51 considered troublesome? Is the difficulty related to the subtleties of connectedness in the definition of a homeomorphism? How would you explain the problem?
5. How can a Möbius strip be constructed from an annulus with a radial cut as in Figure 4 on p. 52? In what way does this experiment test the question about the amount of edge involved in forming a Möbius band?
6. How can a Möbius strip be constructed from a disk with a radial cut to its center? What are the lengths of the joined and unjoined edges of the resulting Möbius band? Let us define the width of a Möbius band to be the length of either of its two opposite edges that are glued together (after adding the half-twist). Let us define its length to be the length of its one and only edge. What is the width to length ratio in this Möbius strip?
7. How can a Möbius band be constructed from a semicircle as in Figure 11 on p. 55? What are the lengths of the joined and unjoined edges of the resulting Möbius strip? What is the width to length ratio in this Möbius strip?
8. How can a Möbius strip be constructed from a 30° sector of a circle? What are the lengths of the joined and unjoined edges of the resulting Möbius strip? What is the width to length ratio in this Möbius band?
9. After reading chapter 4, can you explain what Barr's text means on p. 52 where it says "some meaningful restrictions must be placed on the Möbius strip, too, as to how much of the edges ought to be joined" and in wondering "if the amount of edge involved can be *increased*"? What does it mean to increase or decrease the amount of edge involved? What are the restrictions alluded to?

[1]Note: Please let me know of any difficulties. There may be a revised version correcting issues if any are found. Compiled by CJ Fearnley. `http://blog.CJFearnley.com`.

[2]`https://www.meetup.com/MathCounts/events/246163335/`

10. On p. 61, Barr concludes chapter 4 by saying "The moral of all this is that when we allow *only one kind* of distortion (bending), unexpected relationships persist. Suspicion arises that with *any* distortion allowed, what persists must be invariant indeed, *and perhaps overlooked before*." How do you interpret this conclusion? What invariants did you infer from reading chapter 4 and any experiments you undertook?

11. In considering the variations on a Möbius band in chapter 4 and any additional thought or model-building experiments you might have undertaken, how should we think about the distortion and connectedness in the joined edge of the strip of paper used to make a Möbius band? How is the homeomorphic property preserved in each of these variations?

12. In considering the sequence of experiments discussed in chapter 4 where Möbius strips with various extents of connectedness are considered, what did you learn about the nature of homeomorphism, topological invariants, and paper representations of topological surfaces?

13. Given the considerations in chapter 4, what subtleties, limitations, and caveats must we heed about the nature of the distortions allowed and the requirements for connectedness and continuity in Stephen Barr's definition of a homeomorphism?

14. In the description of the Klein bottle on pp. 34–35, the order of joining the opposite edges is discussed. Why is it that it seems impossible to make the half-twist Möbius join first? What is it about the order of operations that makes doing the untwisted join first easier to imagine?

15. In Fig. 22 and 23 on p. 37 and the surrounding text, a way to build a symmetrical Klein bottle is described. How can you build it with paper? How should we interpret this model given the statements in the text on p. 38 that "the surface passes through itself" and "That is to say, in intersecting, neither plane interrupts the continuity of the other"?

16. How can one see and understand the nonorientability of the Klein bottle? How can you appreciate its nonorientability in a paper model? Can we understand the nonorientability from the connectedness diagram in figure 18 on p. 34? Does the text on pp. 62–63 clarify the matter?

17. On p. 63, the book asks "What happens when a Klein bottle is cut in two?" What do you think? How many cases need to be considered?

18. How can you interpret the model in Fig. 5 on p. 65 as a Klein bottle? If we regard the joint CC' as a cut, how can you interpret the resulting model? How would you describe that cut and its results?

19. How can you interpret the model in Fig. 10 on p. 68 as a Klein bottle? If we regard the joint between the edges AB' and $A'B$ as a cut, how can you interpret the resulting model? How would you describe that cut and its results?

20. In considering the conical Möbius band and the Klein bottle and two of its dissections at the beginning of chapter 5, what observations, realizations, understandings, and insights have you had about the nature of topology, homeomorphism, orientability, and the connectedness of topological surfaces?