Supercircles: Expanding Buckminster Fuller's Foldable Circle Models

Chris Fearnley
cjf@CJFearnley.com

Jeannie Moberly
moberlypj@yahoo.com

240 Copley Road
Upper Darby, PA 19082, USA

Paper: http://www.CJFearnley.com/supercircles.pdf

Slides: http://www.CJFearnley.com/supercircles.slides.01.pdf

Abstract

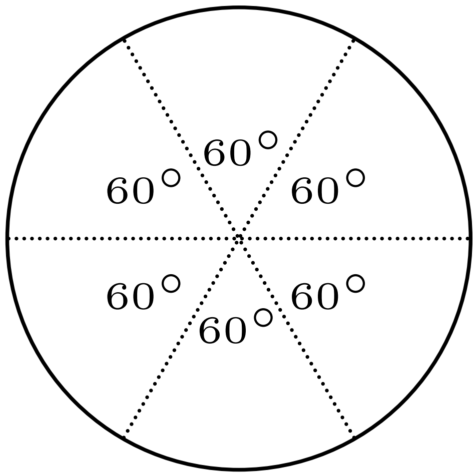
We tell our story of an investigation to discover how to expand Buckminster Fuller's foldable circle, model-building method to other great circle geometries. We discovered that Fuller's method can be generalized by the introduction of what we call supercircles (buildable constructs that are essentially circular, but have more than 360 degrees of arc). We outline our techniques for building the models and identify relationships to the relevant elements of polyhedron geometry, spherical trigonometry, group theory, combinatorics and graph theory. In the process we have identified a number of interesting mathematical questions which may lead to a theory of great circle foldabilities.

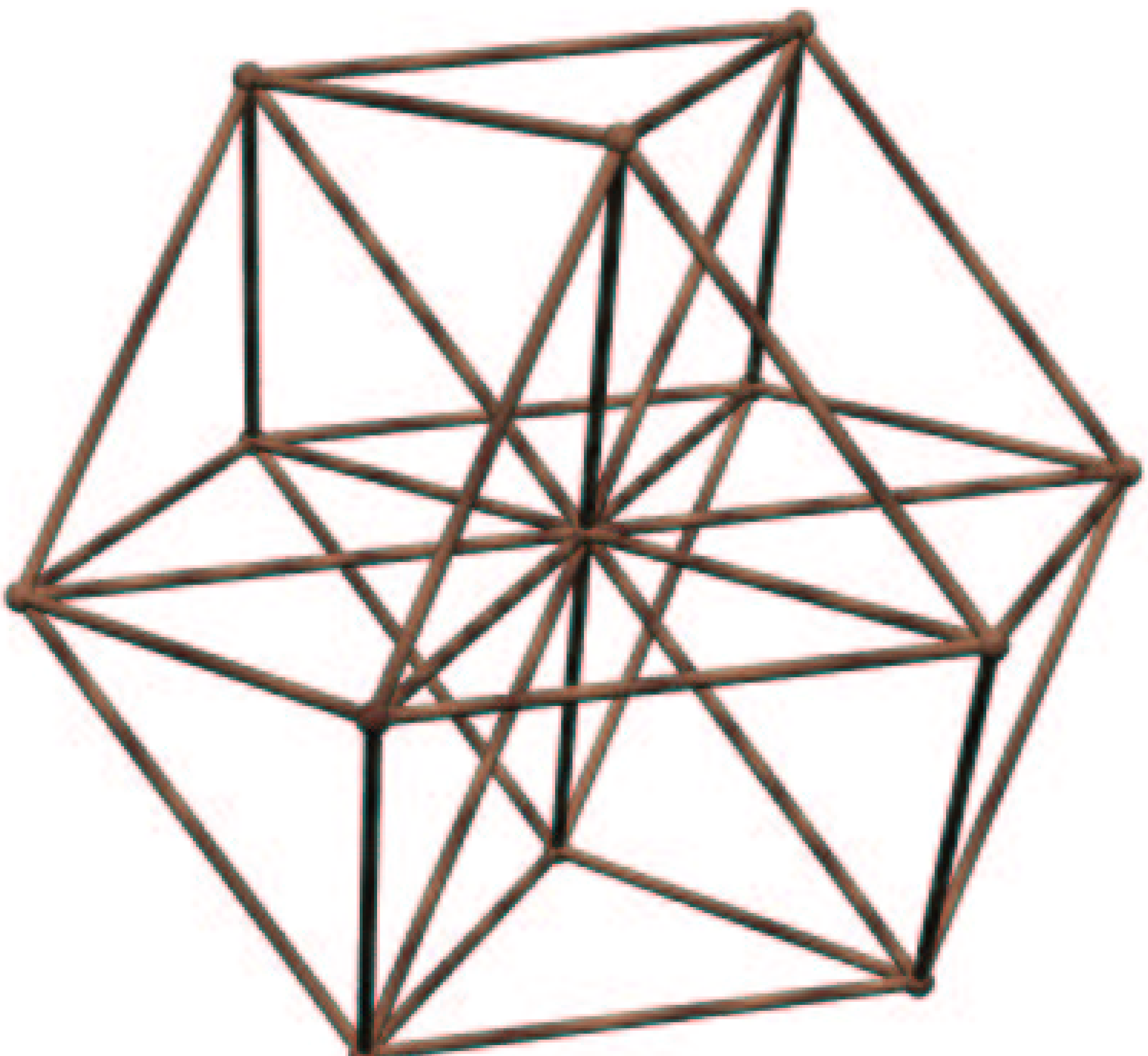
On 12 July 2004 the U.S. Postal Service issued a commemorative postage stamp honoring Buckminster Fuller the legendary American inventor, architect, engineer, designer, geometrician, cartographer and philosopher.

The Cuboctahedron: Fuller's Vector Equilibrium (VE)

- 12 radii of unit length
- 24 circumferential edges of unit length
- 8 triangular and 6 square faces or openings
- 4 hexagonal cross-section planes
- A semi-regular or Archimedean polyhedra
- A **quasi-regular polyhedra**: formed by connecting the vertices where the common mid-sphere of two regular polyhedra intersect
- A stick model with flexible joints is a "jitterbug" (undergoes a dynamic transformation like a dance)
- The closest packing of spheres around a nuclear ball can form the VE shape; indeed, when the spheres form a VE, they aggregate in successive circumferential layers each of which is a VE
- Two antipodal triangles define an axis of spin which generates the spherical VE
- The VE is the simplest (least number of modules) foldable great circle geometry that Fuller described

Folding the VE





“The most important fact about Spaceship Earth: an instruction manual didn’t come with it.”

— R. Buckminster Fuller

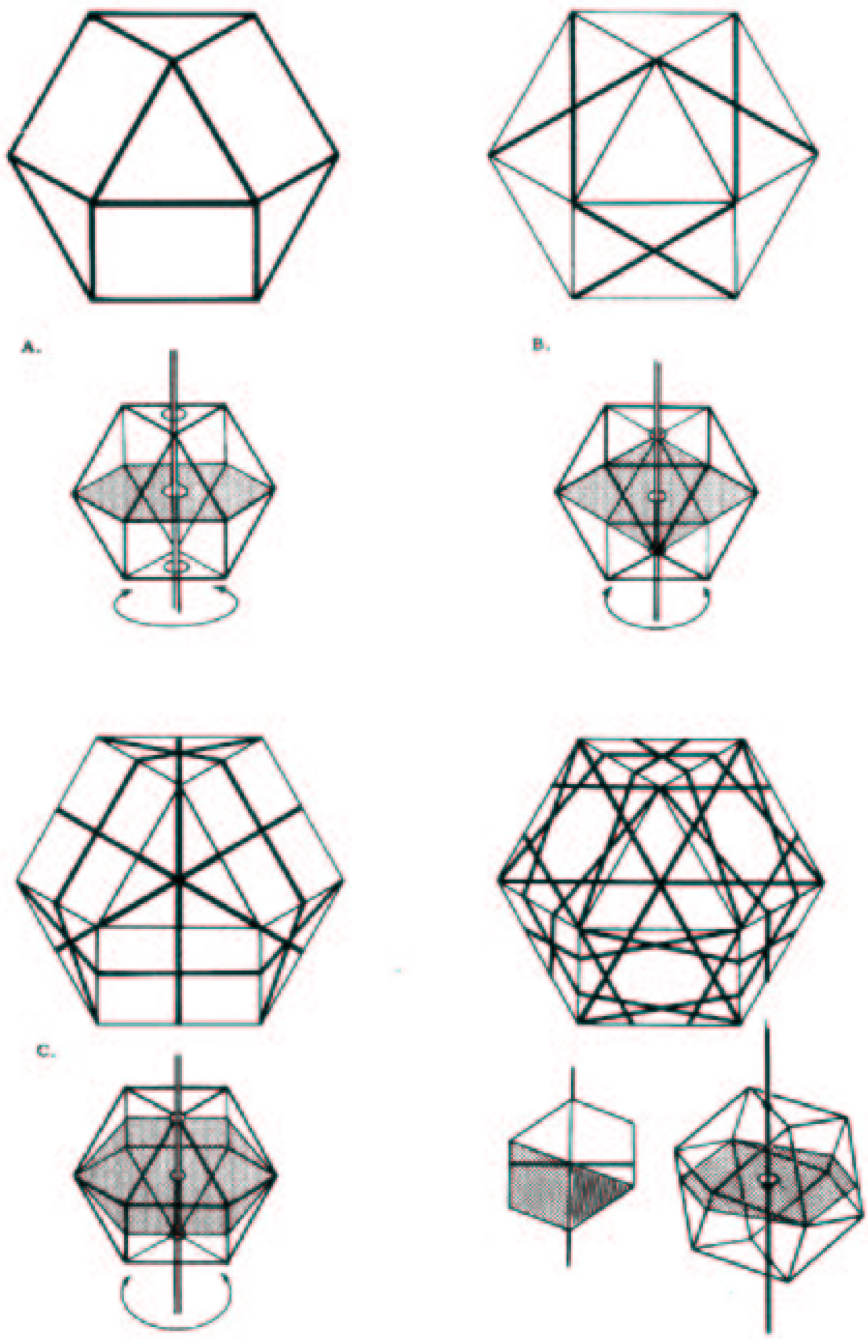


Fig. 450.11A Axes of Rotation of Vector Equilibrium:

A. Rotation of vector equilibrium on axes through centers of opposite trianglar faces defines four equatorial great-circle planes.
B. Rotation of the vector equilibrium on axes through centers of opposite square faces defines three equatorial great-circle planes.
C. Rotation of vector equilibrium on axes through opposite vertexes defines six equatorial great-circle planes.
D. Rotation of the vector equilibrium on axes through centers of opposite edges defines twelve equatorial great-circle planes.

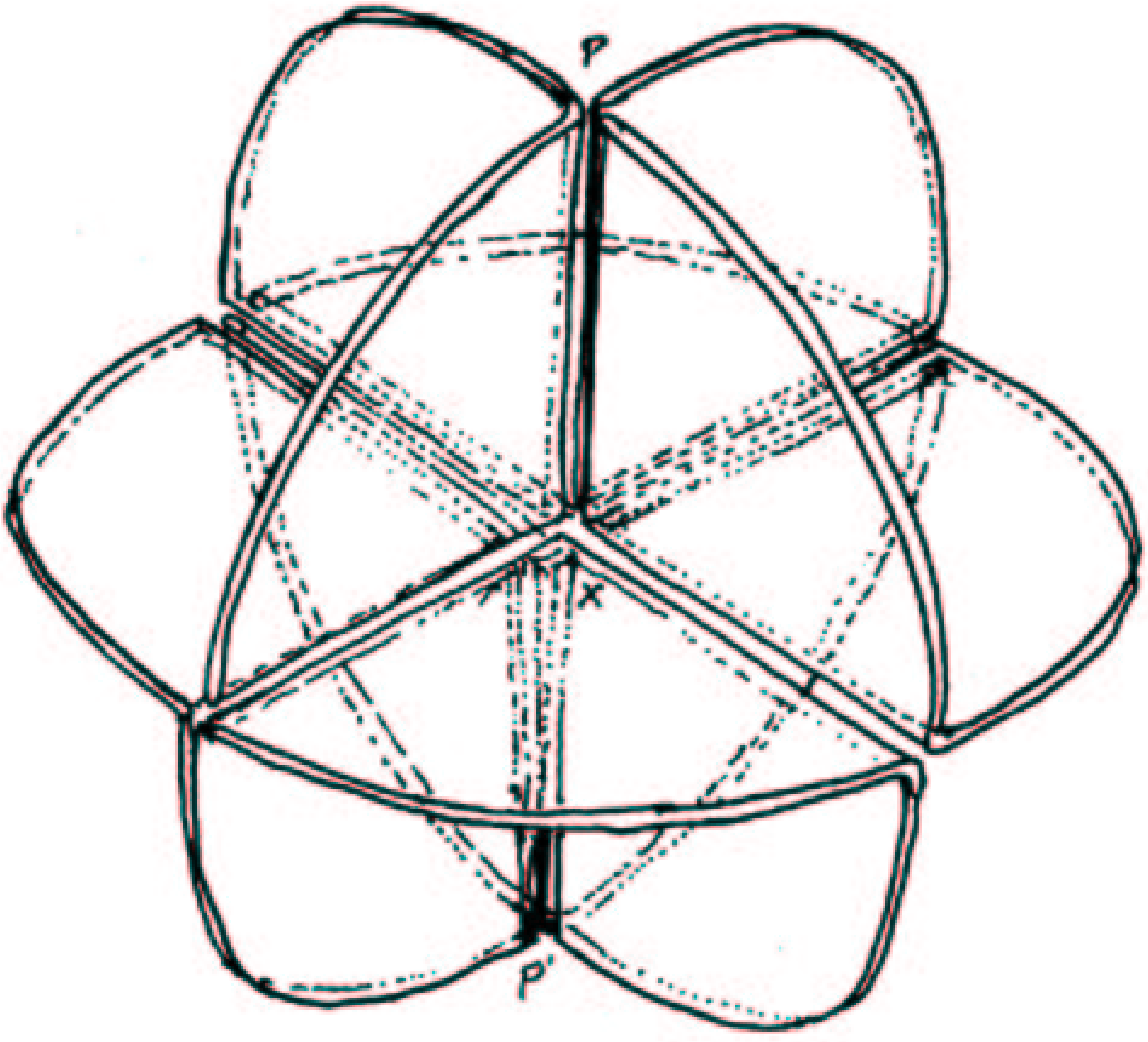


Fig. 835.10 Six Great Circles Folded to Form Octahedron.

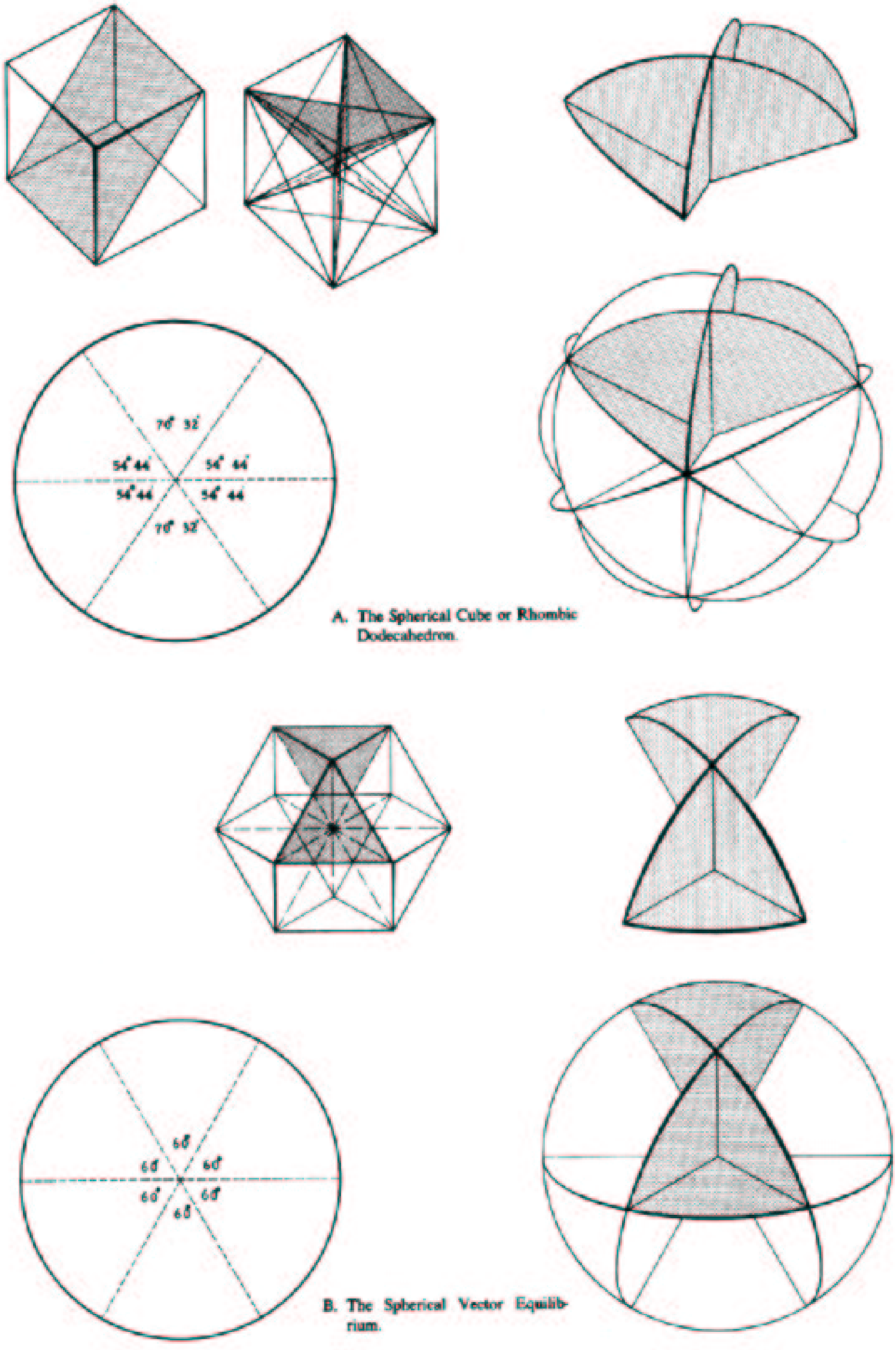


Fig. 455.11 Folding of Great Circles into Spherical Cube or Rhombic Dodecahedron and Vector Equilibrium: Bow-Tie Units:

A. This six-great-circle construction defines the positive-negative spherical tetrahedrons within the cube. This also reveals a spherical rhombic dodecahedron. The circles are folded into "bow-tie" units as shown. The shaded rectangles in the upper left indicates the typical plane represented by the six great circles.
B. The vector equilibrium is formed by four great circle folded into "bow-ties." The sum of the areas of the four great circles equals the surface area of the sphere. ($4\pi r^2$).

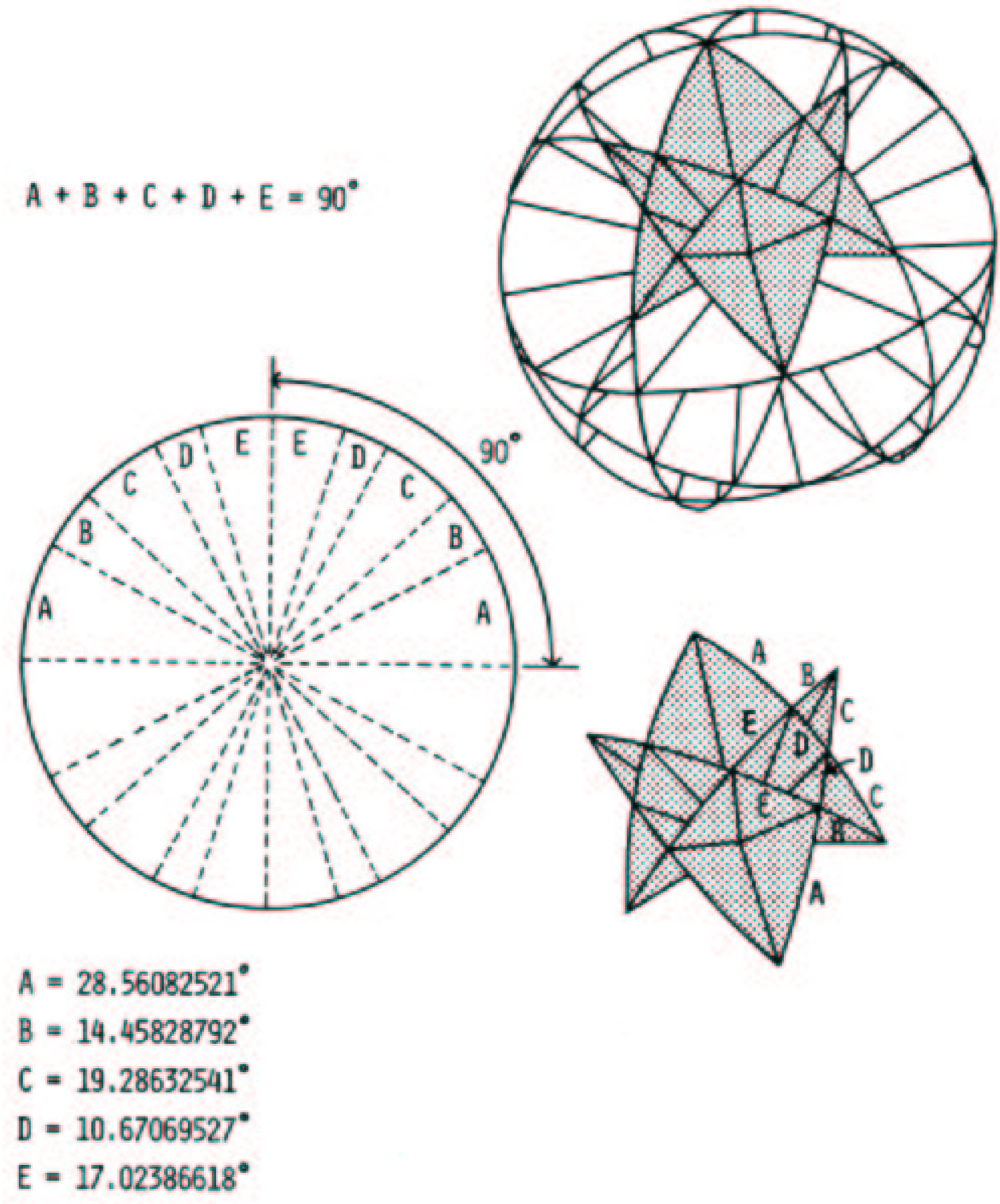


Fig. 450.10 The 12 Great Circles of the Vector Equilibrium Constructed from 12 Folded Units (Shwon as Shaded).

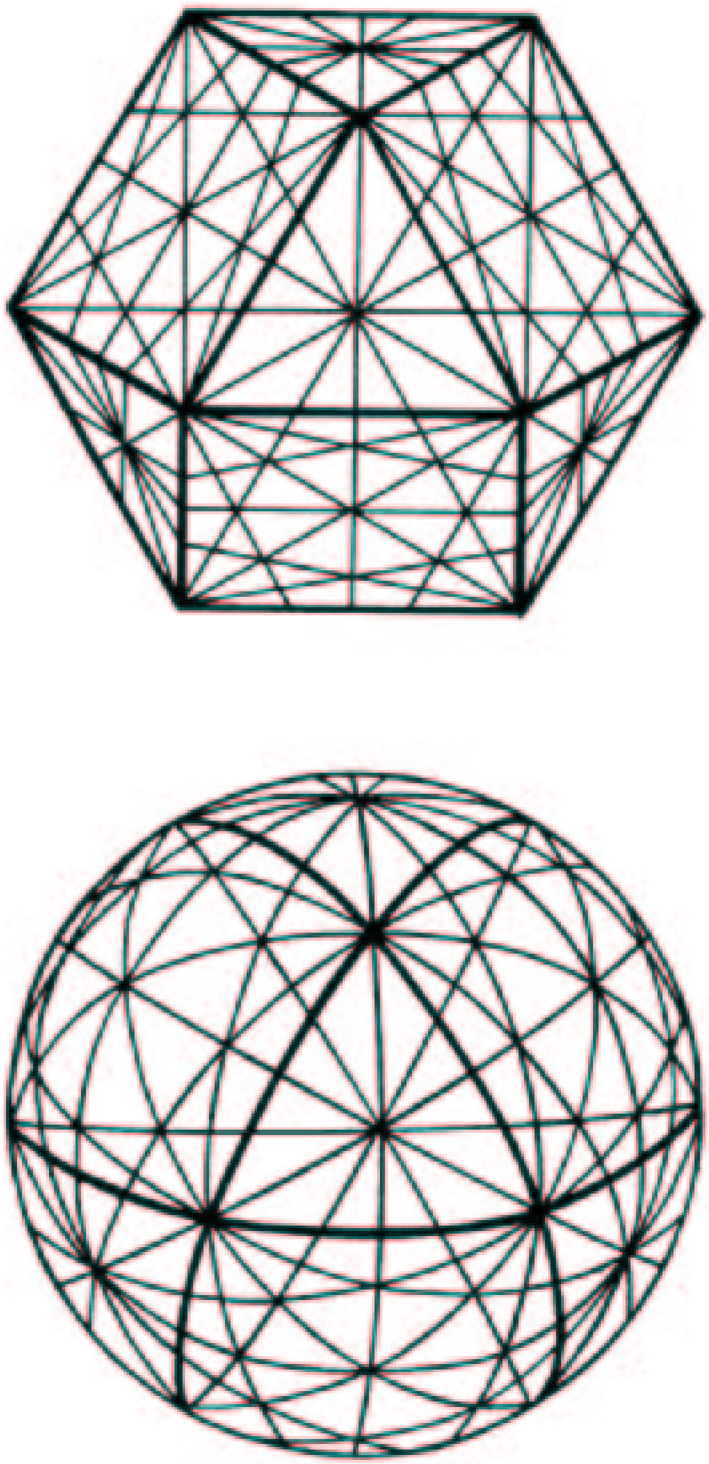
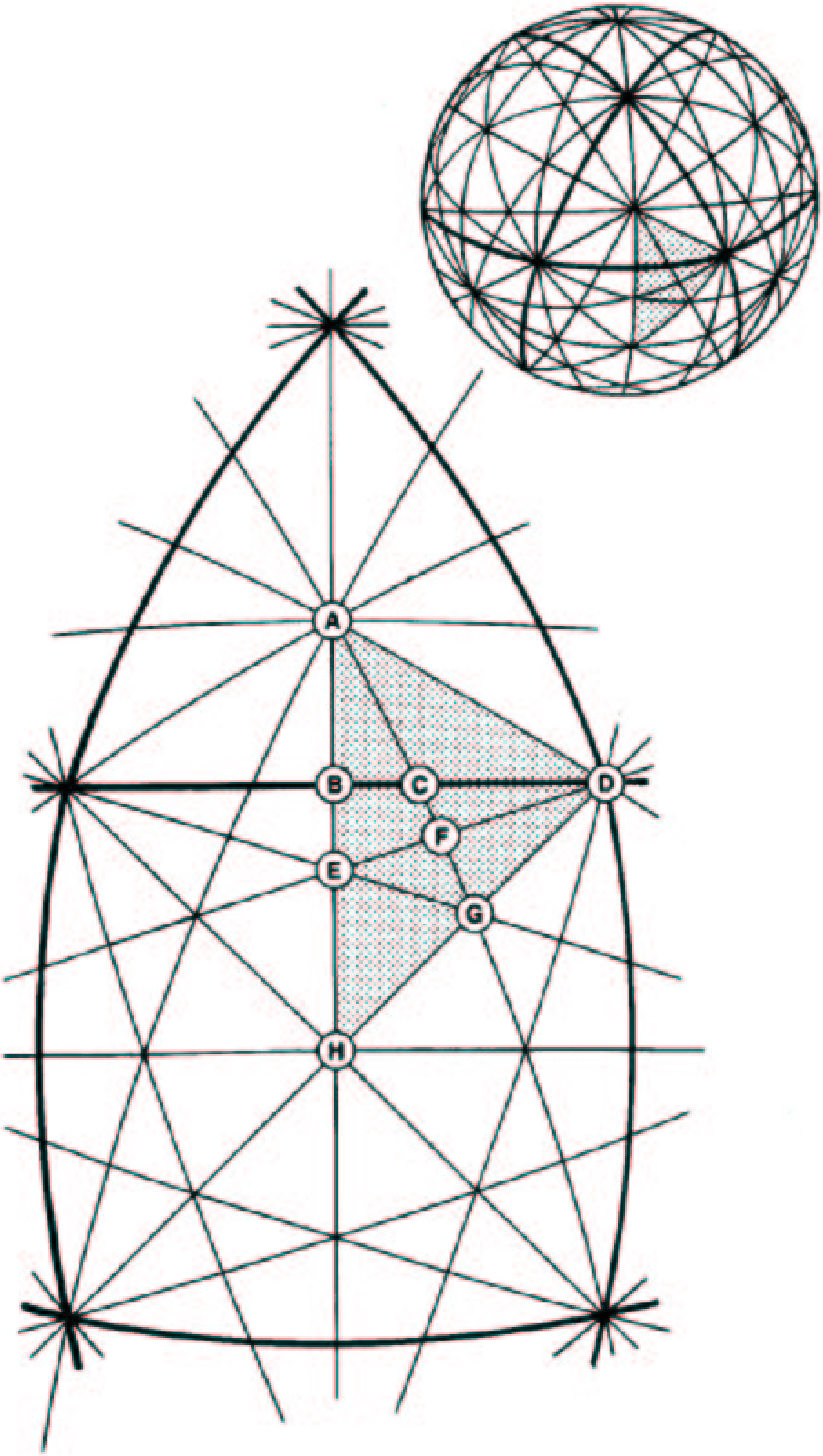


Fig. 450.11B Projection of 25 Great-Circle Planes in Vector Equilibrium Systems: The complete vector equilibrium system of 25 great-circle planes, shown as both a plane faced-figure and as the complete sphere (3 + 4 + 6 + 12 = 25). The heavy lines show the edges of the original 14-faced vector equilibrium.



CENTRAL ANGLES		
19.47122063°	AB	19° 28' 16.394"
35.26438968	AD	35 15 51.803
22.20765430	AC	22 12 27.555
10.89339465	BC	10 53 36.221
19.10660535	CD	19 06 23.779
10.02498786	BE	10 01 29.956
6.35317091	CF	6 21 11.415
14.45828792	EF	14 27 29.837
17.02386618	FD	17 01 25.918
19.28632541	EG	19 17 10.771
10.67069527	FG	10 40 14.503
25.23940182	EH	25 14 21.847
26.56505118	HG	26 33 54.184
18.43494882	GD	18 26 5.816
31.48215410	DE	31 28 55.755
30.	BD	30 00 00
45.	DH	45 00 00
54.73561031	AH	54 44 8.197

FACE ANGLES		
30.°	BAC	30° 00' 00.000"
30.	CAD	30 00 00.000
90.	ABC	90 00 00.000
61.87449430°	ACB	61 52 28.179
118.1255057	ACD	118 7 31.821
35.26438968	ADC	35 15 51.803
90.	EBC	90 00 00.000
118.1255057	BCF	118 7 31.821
73.22134512	BEF	73 13 16.842
80.40593179	CFE	80 24 21.354
61.87449430	FCD	61 52 28.179
19.47122063	CDF	19 28 16.394
99.59406821	CFD	99 35 38.646
73.22134512	HEG	73 13 16.842
65.90515745	EGH	65 54 18.567
45.	EHG	45 00 00.000
99.59406821	EFG	99 35 38.646
33.55730977	FEG	33 33 26.315
48.18968511	FGE	48 11 22.866
80.40593179	GFD	80 24 21.354
35.26438969	FDG	35 15 51.803
65.90515745	FGD	65 54 18.567

Fig. 453.01 Great Circles of Vector Equilibrium Define Lowest Common Multiple Triangle: 1/48th of a Sphere: The shaded triangle is 1/48th of the entire sphere and is the lowest common denominator (in 24 rights and 24 lefts) of the total spherical surface. The 48 LCD triangles defined by the 25 great circles of the vector equilibrium are grouped together in whole increments to define exactly the spherical surface areas, edges, and vertexes of the spherical tetrahedron, spherical cube, spherical octahedron, and spherical rhombic dodecahedron. The heavy lines are the edges of the four great circles of the vector equilibrium. Included here is the spherical trigonometry data for this lowest-common-denominator triangle of 25-great-circle hierarchy of the vector equilibrium.

"Mathematics is the science of
structure and pattern in general."

Massachusetts Institute of Technology,
Department of Mathematics

Our Main Results

Our Project: We started with a question: How to build a model of the 25 great circles of the VE with foldable circles? Since Fuller did not demonstrate how to build such a model, we thought this project might result in some interesting discoveries (at least we'd learn something new!).

Result 1: A model of the 25 great circles of the VE **cannot** be constructed from 25 identical foldable circle modules.

Result 2: One can build a model of the 25 great circles of the VE with 24 identical foldable modules, each a "supercircle" of 375°.

Definition: a **supercircle** is a geometrical construct derived from a circle by cutting along a radius and splicing in a sector. The resulting figure has more than 360° of arc.

"[Proof is no more than] the testing of the products of our intuition ... Obviously we don't possess, and probably will never possess, any standard of proof that is independent of time, the thing to be proved, or the person or school of thought using it. And under these conditions, the sensible thing to do seems to be to admit that there is no such thing, generally, as absolute truth in mathematics, whatever the public may think."

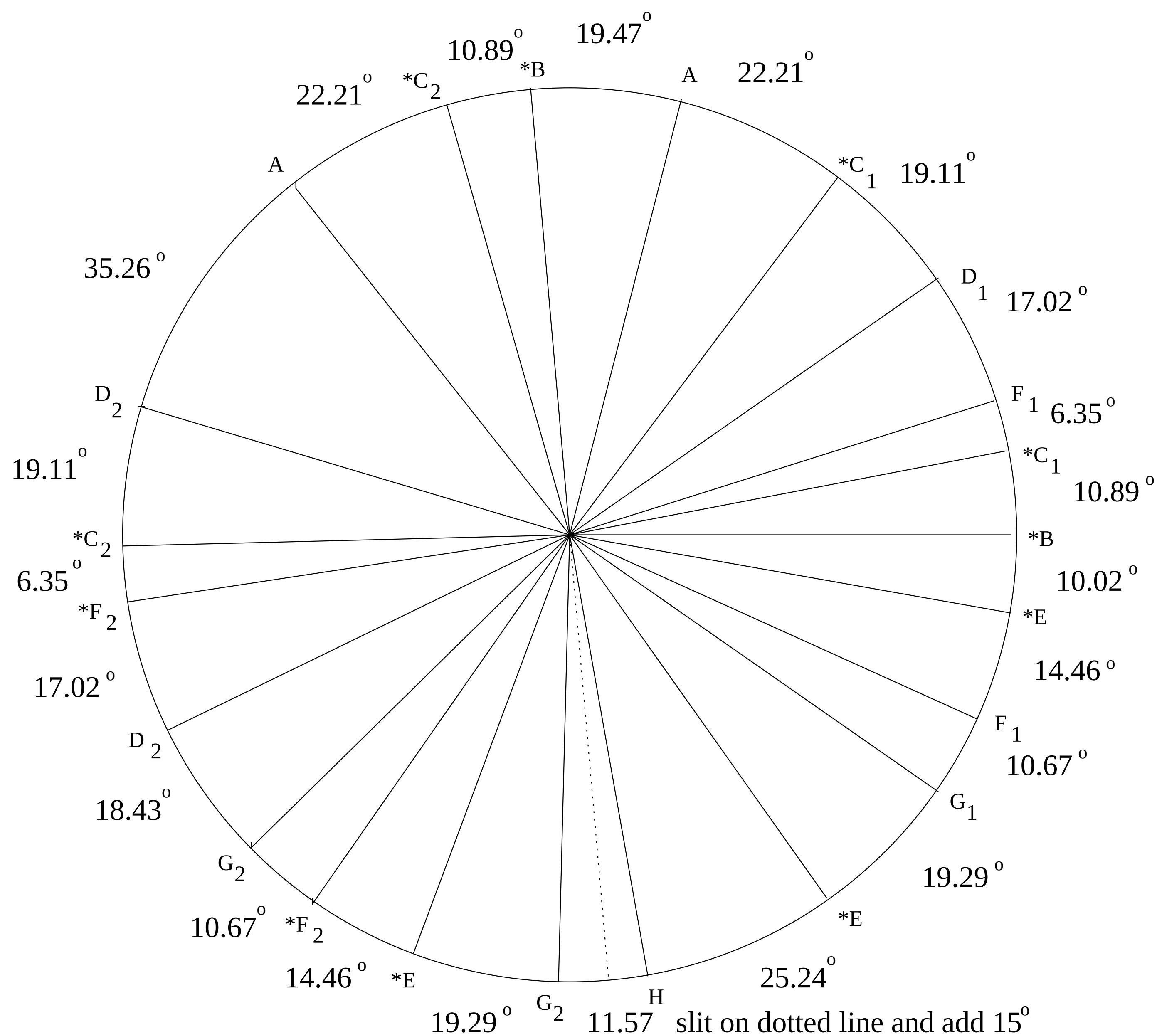
— Raymond L. Wilder

Result 1: A model of the 25 great circles of the VE cannot be constructed from 25 identical foldable circle modules.

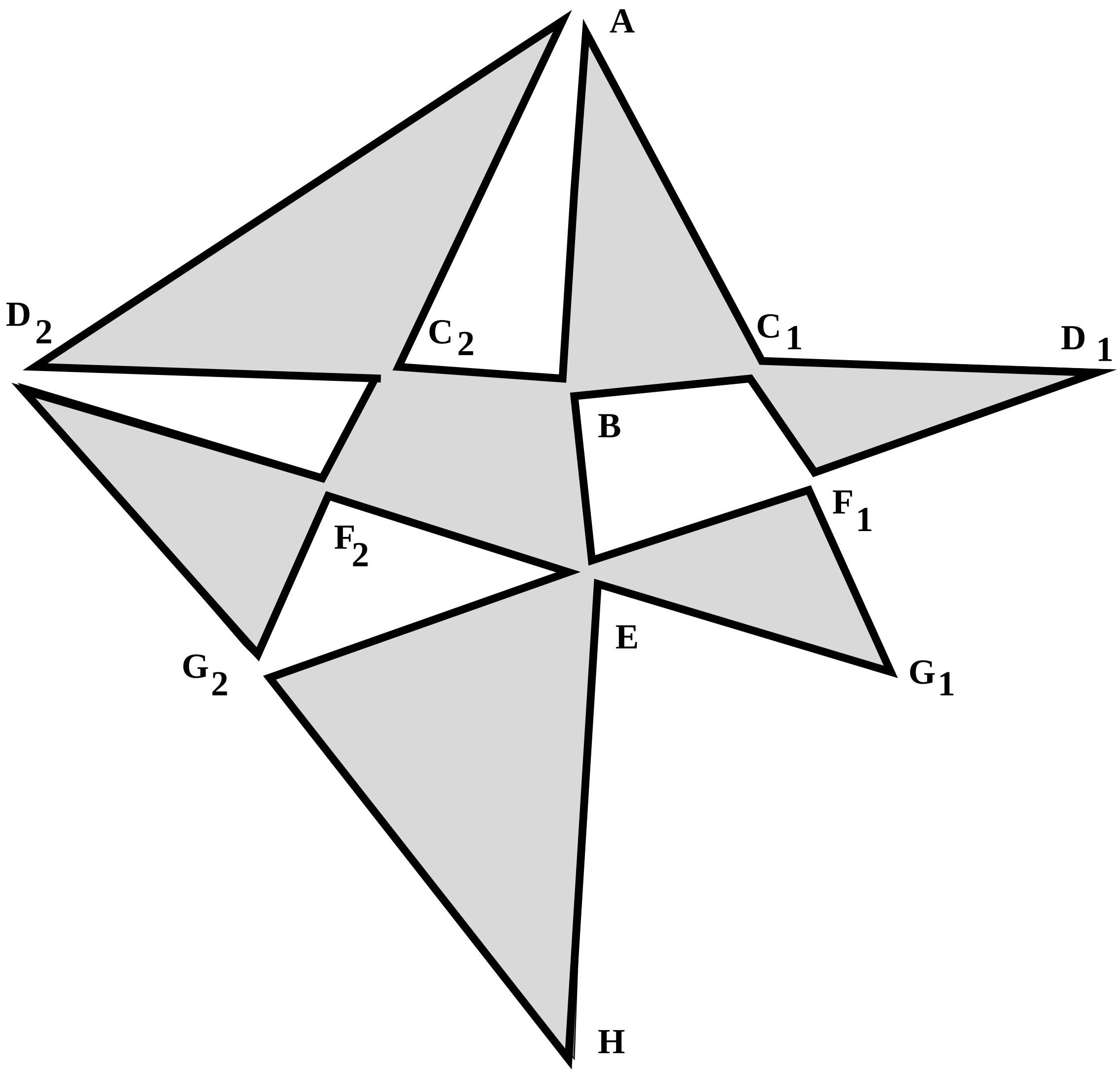
Proof: Consider the $\frac{1}{48}$ LCD triangle (or Schwartz triangle which by repeated reflection in their sides covers the sphere a finite number of times) of the 25 great circles of the VE. Each arc interior to the LCD triangle occurs 48 times in the 25 great circles. But each arc on the boundary or edge is shared by two LCD triangles and only occurs 24 times. Since each module must be identical for our construction, we need to associate each of the 25 great circles (pigeonholes or boxes) with each of the 24 distinct occurrences of the boundary arcs (pigeons or objects). That is impossible (pigeonhole principle or Dirichlet's box principle).

Result 2: One can build a model of the 25 great circles of the VE with 24 identical foldable modules, each a "supercircle" of 375°.

Proof: By construction as follows ... .



This shows one of 24 supercircle modules for the spherical cuboctahedron. This supercircle contains 375° when a 15° sector is inserted at the dotted line. Score asterisk (*) lines below and fold outward. Score non lines on the topside and fold inward. Attach A to A, $*C_1$ to $*C_1$, and so on for like labeled vertices.*



This shows the result of folding the supercircle in figure 1. Connect inwardly within the module B (midpoint of edge), C_1, C_2, E and F_2. Connect F_1 outwardly within the module. Connect A (center of triangle), D (vertex), G_1, G_2, and H (center of square) to adjoining modules. Notice how in this symmetrical module spaces on one side are inside (shaded) and outside (unshaded) on the other, positive and negative space. 24 of these folded supercircle modules composes the foldable 25 great circle model of the spherical cuboctahedron (or VE).

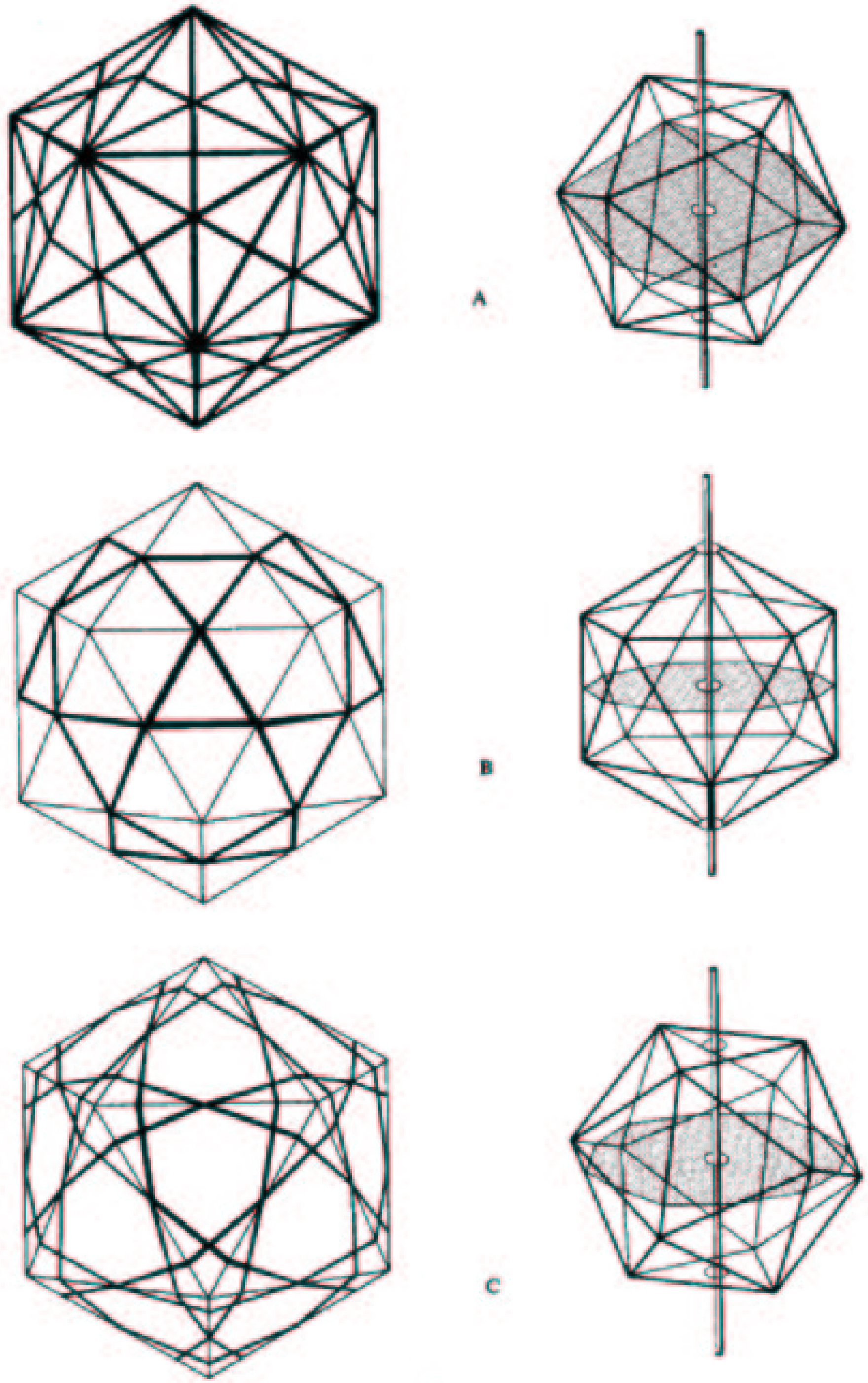


Fig. 457.30A Axes of Rotation of Icosahedron:

A. The rotation of the icosahedron on axes through midpoints of opposite edges define 15 great-circle planes.
B. The rotation of the icosahedron on axes through opposite vertexes define six equatorial great-circle planes, none of which pass through any vertexes.
C. The rotation of the icosahedron on axes through the centers of opposite faces define ten equatorial great-circle planes, which do not pass through any vertexes.

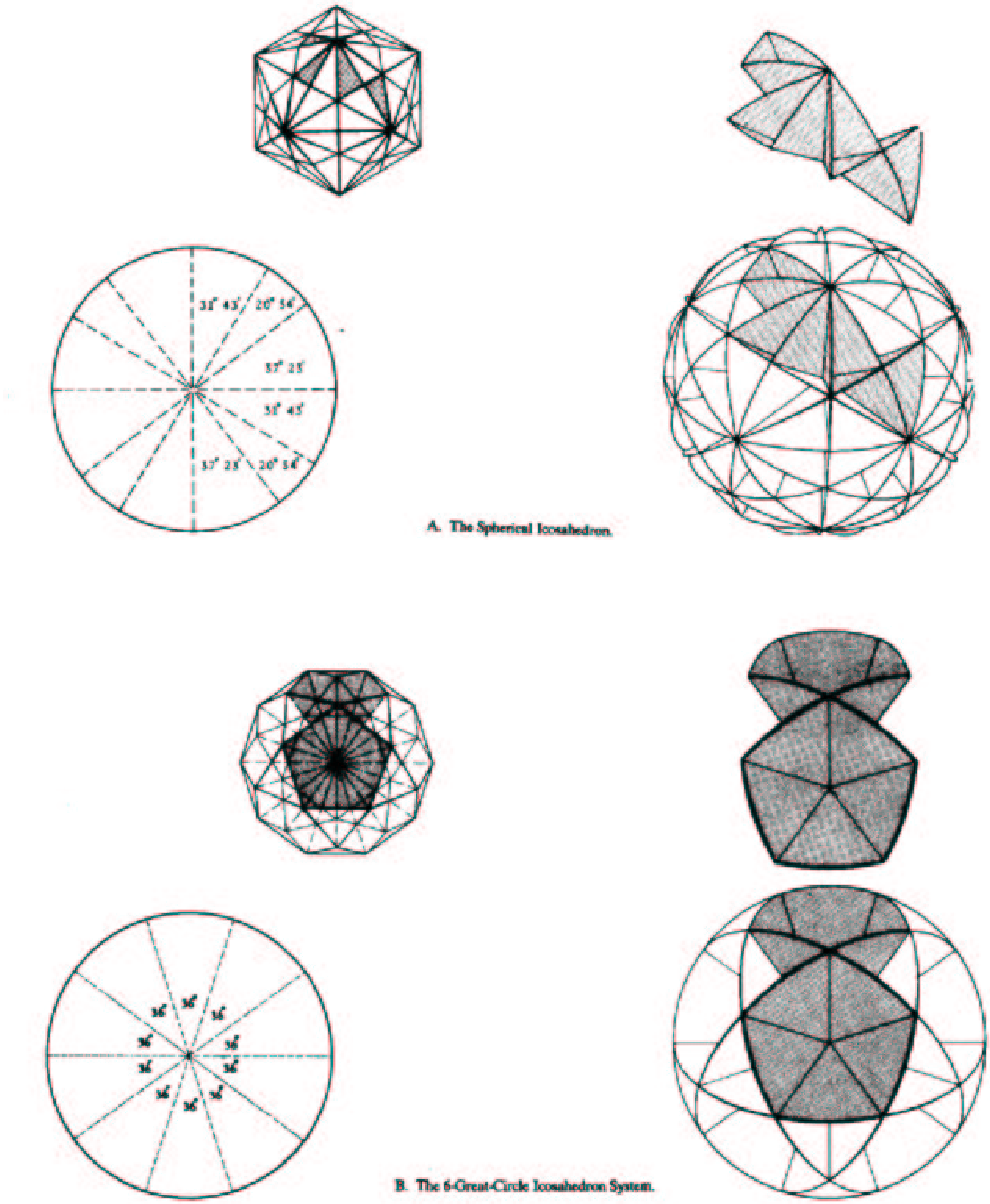


Fig. 458.12 Folding of Great Circles into the Icosahedron System:

A. The 15 great circles of the icosahedron folded into "multi-bow-ties" consisting of four tetrahedrons each. Four times 15 equals 60, which is 1/2 the number of triangles on the sphere. Sixty additional triangles inadvertently appear, revealing the 120 identical (although right- and left- handed) spherical triangles, which are the maximum number of like units that may be used to subdivide the sphere.
B. The six great-circle icosahedron system created from six pentagonal "bow-ties."

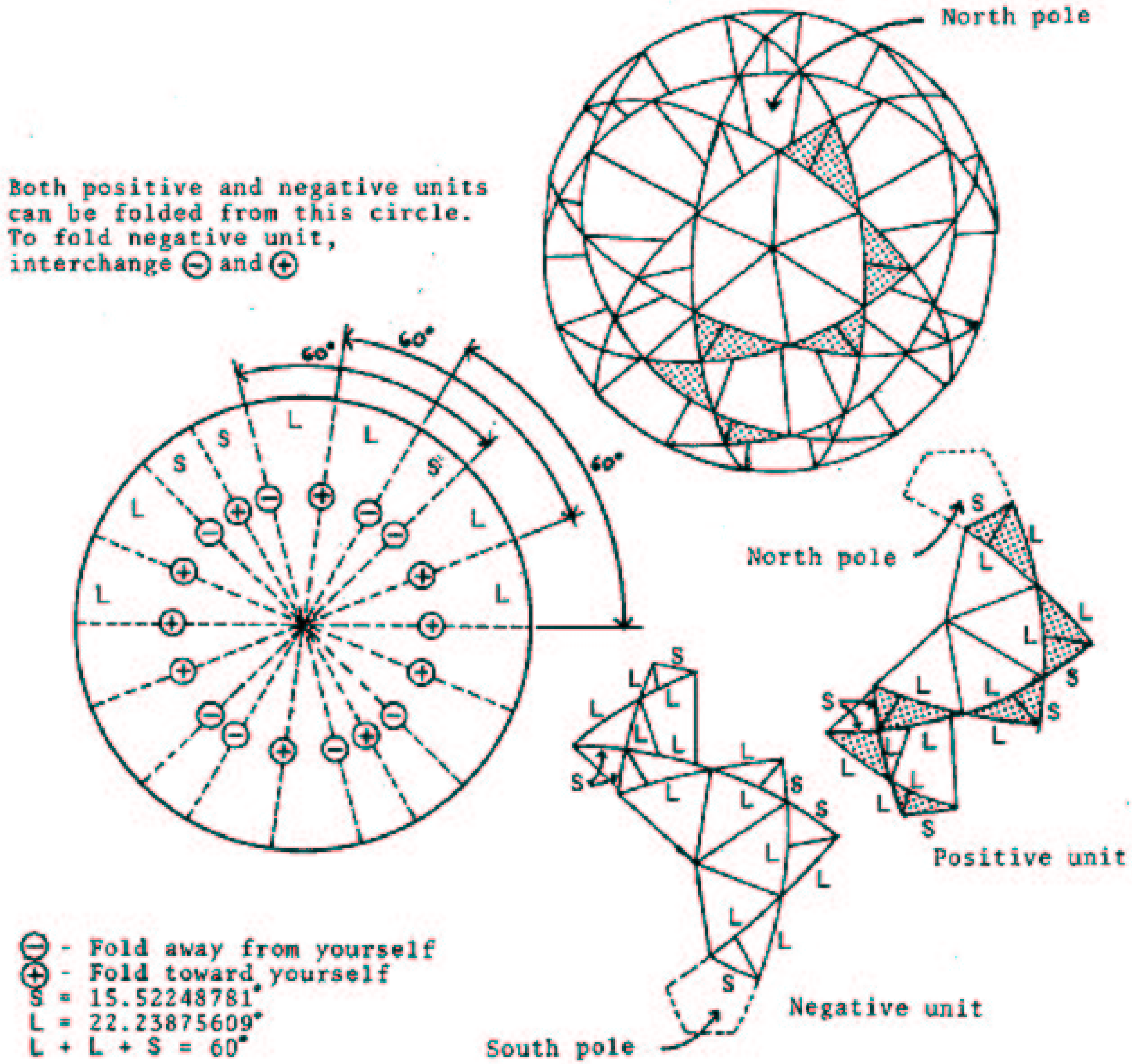


Fig. 455.20 The 10 great circles of the Icosahedron Constructed from 10 folded units (5 positive units + 5 negative units).

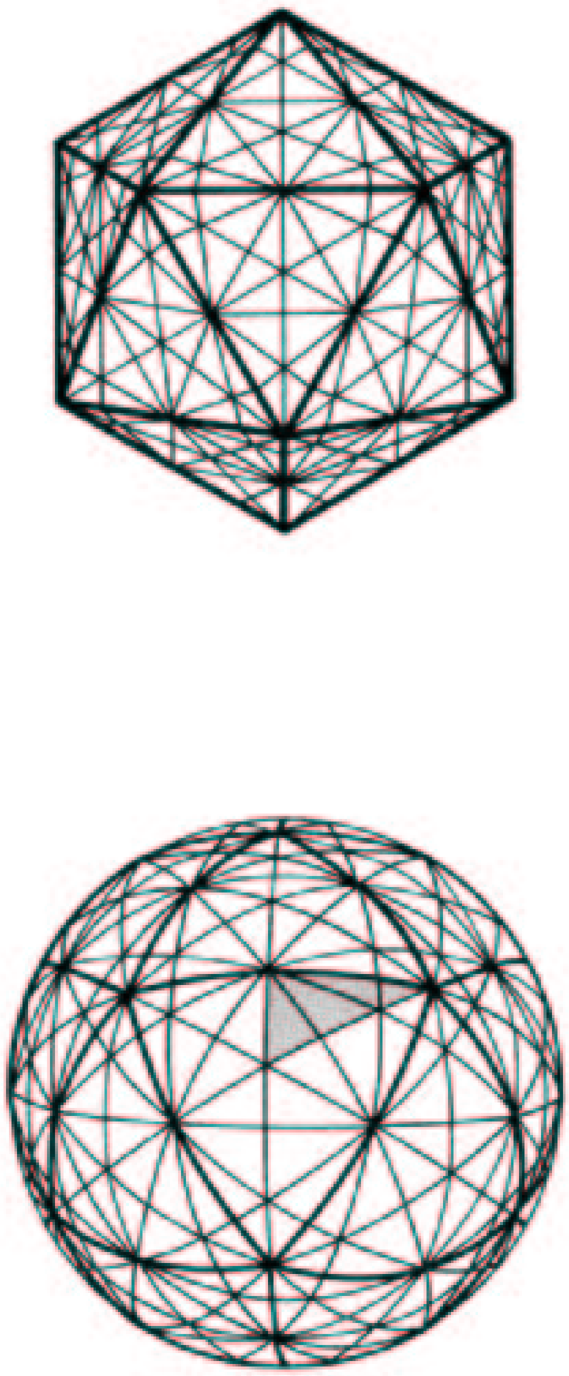


Fig. 457.30B Projection of 31 Great-Circle Planes in Icosahedron System: The complete icosahedron system of 31 great-circle planes shown with the planar icosahedron as well as true circles on a sphere (6+10+15=31). The heavy lines show the edges of the original 20-faced icosahedron.

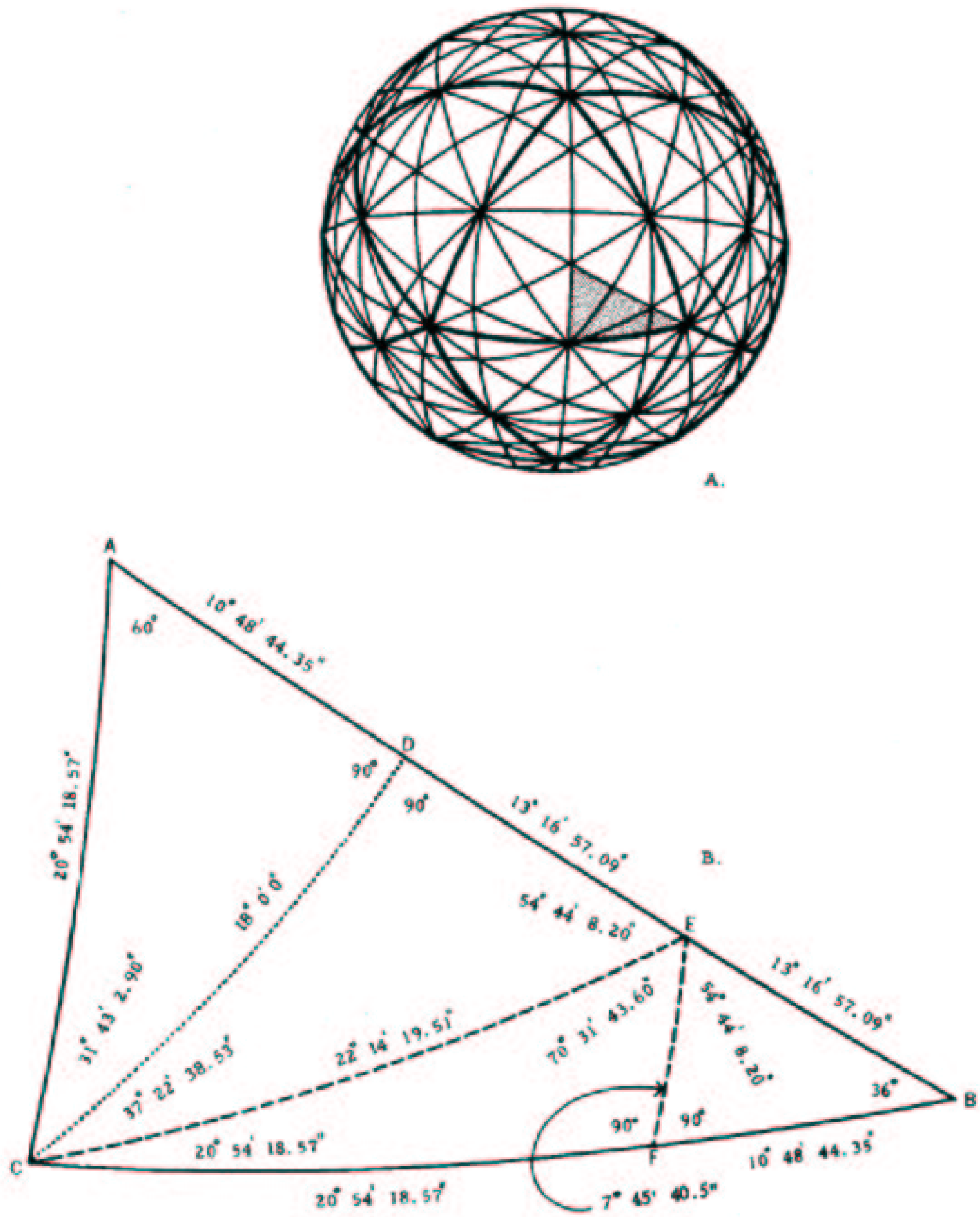


Fig. 901.03 Basic Right Triangle of Geodesic Sphere: Shown here is the basic data for the 31 great circles of the spherical icosahedron, which is the basis for all geodesic dome calculations. The basic right triangle as the lowest common denominator of a sphere's surface includes all the data for the entire sphere. It is precisely 1/120th of the sphere's surface and is shown as shaded on the 31-great-circle- sphere (A). An enlarged view of the same triangle is shown (B) with all of the basic data denoted. There are three different external edges and three different internal edges for a total of six different edges. There are six different internal angles other than 60° or 90°. Note that all data given is spherical data, i.e. edges are given as central angles and face angles are for spherical triangles. No chord factors are shown. Those not already indicated elsewhere are given by the equation 2 sin(theta/2), where theta is the central angle. Solid lines denote the set of 15 great circles. Dashed lines denote the set of 10 great circles. Dotted lines denote the set of 6 great circles.

Model-Building Tips

- *How to join vertices together?*
  - *Fuller recommended bobby pins*
    - *Advantages*
      - *Tension pulls the model together*
      - *Easy to take apart and put back together*
    - *Disadvantages*
      - *heavy and difficult to use at vertices whose degree is greater than 4*
  - *Jeannie has tried sewing with thread, slits and tabs, and yellow wood glue.*
  - *Better methods could be developed (perhaps, bobby pins with 3–8 prongs).*
- *Materials: Tyvek, paper, cardboard. Stiffer circles tend to bend less creating a more perfect great circle appearance, but they are harder to fold.*
- *When a model is being put together it has a tendency to be slack until the last few bobby pins are inserted. The tension then pulls the model taut and the slack parts stretch out. This shows that the model requires tensional connections (like one of Fuller's tensegrities).*

Summary of Fuller's 7 Foldable Models

Modules	*GC*[a]	*SE*[b]	*Arcs in a module*	*Derivation*[c]
3	*3*	*0*	$4 \times 90^\circ = 360^\circ$	*octa vertices* / *Spherical Octahedron*
4	*4*	*0*	$6 \times 60^\circ = 360^\circ$	*octa faces or triangular VE faces* / *Spherical Cuboctahedron (VE)*
6	*6*	*0*	$2 \times (70^\circ 32' + 2 \times 54^\circ 44') = 360^\circ$	*VE edges*
6	*6*	*0*	$10 \times 36^\circ$	*icosa vertices* / *Spherical Icosidodecahedron*
10	*10*	*0*	$6 \times (15.522^\circ + 2 \times 22.238^\circ) = 360^\circ$	*icosa faces*
12	*12*	*0*	$4 \times (28.561^\circ + 14.458^\circ + 19.286^\circ + 10.671^\circ + 17.024^\circ) = 360^\circ$	*VE vertices or octa edges*
15	*15*	*0*	$4 \times (31^\circ 42' + 20^\circ 54' + 37^\circ 23') = 360^\circ$	*icosa edges*

[a]Number of Great Circles
[b]Supercircle Excess: amount of arc in excess of 360°
[c]Equators of a Spin Axis for antipodal topological elements

Modules	GC[a]	SE[b]	Arcs in a module	Derivation[c]
Our Foldable Supercircle Models				
8	*7*	90°	$6\times45°+3\times60° = 450°$	3 + 4 = 7 *great circles of the VE*
12	*13*	30°	$4(35°16' + 54°44') + 2(45° + 30° + 19°28'16.394'') = 390°$	3 + 4 + 6 = 13 *great circles of the octa*
24	*25*	15°	*14 arc lengths: 8 of which occur twice* = 375°	3 + 4 + 6 + 12 = 25 *great circles of the VE*
30	*31*	12°	*9 arc lengths: 3 of which occur four times and 6 of which occur twice* = 372°	6 + 10 + 15 = 31 *great circles of the icosa*
Selected Other Foldable Supercircle Models				
3 + 6 = 9, 4 + 6 = 10, *etc. GCs of the VE or octa*				
12 + 24 + 12 + 24 = 96 *secondary GCs of the VE*				
121 *secondary great circles of the icosa*				

[a]Number of Great Circles
[b]Supercircle Excess: amount of arc in excess of 360°
[c]Equators of a Spin Axis for antipodal topological elements

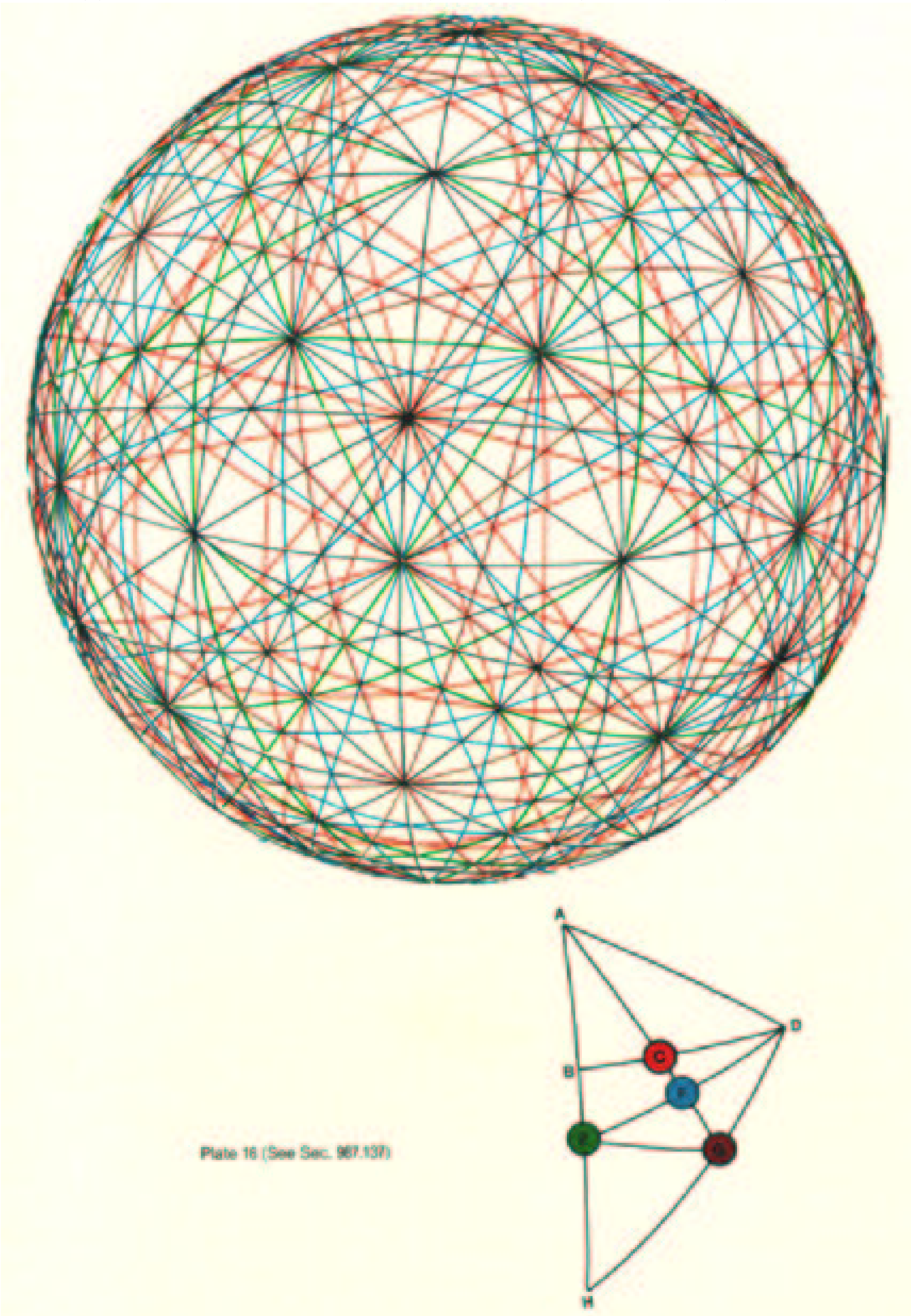


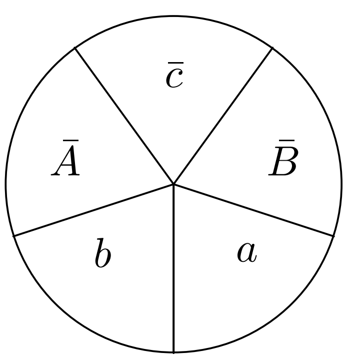
Plate 16 (See Sec. 987.137)

Methods to Generate Great Circle Models

- *Fuller's equators of spin technique (rotational symmetry)*
- *Fuller's secondary equators of spin technique (rotational symmetry)*
- *Subdivide semi-regular and uniform polyhedra into characteristic **012** triangles (joining vertex to midpoint of edge to midpoint of face) then apply above techniques. Other subdivision techniques can be imagined.*
- *Apply these techniques to other semi-regular (Archimedean) polyhedra or Uniform polyhedra (identical vertices)*
- *Any Schwartz triangle*
- *Drawing "lines" (great circles) connecting two points on any spherical model (Coxeter)*
- *Random or statistical techniques*

Spherical Trigonometry

1. *A great circle has* $360°$.
2. *The sum of the spherical angles around a point totals* $360°$.
3. *Arc measure in a great circle is in degrees (or radians) and is given by the central angle measure.*
4. *The measure of a spherical angle is given by the measure of the arc cut off by the sides of the angle at its equator.*
5. *Law of sines:*
$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$
6. *Law of cosines (sides): three formulas by permuting sides*
$$\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos A.$$
7. *Law of cosines (angles): three formulas by permuting angles*
$$\cos A = -\cos B \cdot \cos C + \sin B \cdot \sin C \cdot \cos a.$$



Napier's Rules

8. *Sine of an unknown part equals the product of the cosines of its two opposite parts.*
$$\sin = \cos \cdot \cos$$
9. *Sine of an unknown part equals the product of the tangents of its two adjacent parts.*
$$\sin = \tan \cdot \tan$$

“Dare to be naïve.”

— R. Buckminster Fuller

Observations and Questions Relating to Graph Theory and Combinatorics

- *In general, each module in a great circle geometry is an eulerian closed path (or chain).*
  - *An eulerian circuit is a closed path in a multigraph (a non-simple graph with no self-loops, but multiple edges between any two nodes are allowed) where each edge is traversed once and only once.*
  - *Euler's Theorem (Königsberg Bridge Problem): A multigraph has an eulerian circuit IFF it is connected (there is a path from any point to any other point in the graph) except for isolated vertices and every vertex has even degree (number of neighbors).*
  - *Clearly, great circle geometries are necessarily connected and have even degree at every vertex.*
  - *Does graph theory have results for partitioning graphs on the surface of a sphere into sets of equal eulerian circuits?*
  - *The 3 great circles of VE or octa and the 7 great circles of the VE or tetra have doubled edges. Does graph theory offer any results for exploring these special cases?*
- *Foldability can be seen as a graph theoretical property with the requirement that a spherical multigraph be partitioned into a disjoint set of (equal) eulerian circuits (the modules). Fuller's foldability property requires, in addition, that the sum of the arc measures in each circuit equal* $360°$.
- *Is there a "spherical" or "polyhedral" graph theory that can explain some of the properties of great circle foldability.*

Other Observations and Questions

- *Do there exist any great circle maps besides the seven given by Fuller that can be constructed with 360° modules?*
- *The number of supercircle modules seems to be exactly one off from the number of great circles in the aggregate models.*
- ***super-duper circles** are great circle models folded from a single module (which will have many spliced-in circles, supercircles, and/or sectors)*
  - *How to build super-duper circle modules? They are really complicated!*
- *If there are only 7 Fuller modules (with exactly 360°), a reason may be found in the theory of polyhedral groups*[1]*. Does anyone see the connection?*
  - *Why? The the number of great circles generated by Fuller's modules are divisors of the number of elements in the polyhedral groups. It is remarkable how frequently naïve numerological arguments turn out to be true in polyhedral geometry!*
  - *Complete Tetrahedral Symmetry Group: Isomorphic to* S_4 *(the symmetric group of order 24).*
  - *Tetrahedral Rotation Symmetry Group: Isomorphic to* A_4 *(the alternating group of order 12).*
  - *Octahedral Rotation Symmetry Group: Isomorphic to* S_4 *(the symmetric group of order 24).*
  - *Icosahedral Rotation Symmetry Group: Isomorphic to* A_5 *(the alternating group of order 60).*

[1] H. S. M. Coxeter, Introduction to Geometry, John Wiley & Sons, Inc., second edition, pp. 270–277.

"A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street."

An Old French Mathematician
quoted by David Hilbert

Toward a Theory of Foldability

- *A Theory of Foldability would explain the properties that we have been discussing as a collection of relationships and clarify the connections with other areas of mathematics. This is a partial list of some of the properties that would need to be explained:*
  - *Which great circle geometries are foldable with perfect* 360° *circles? Just the 7 that Fuller found?*
  - *How to explain the doubled-edges in some of the models?*
  - *Are all sets of great circles foldable? Is the supercircle method general purpose and capable of folding any such set?*
  - *Can foldability be seen as a continuum from individual arcs to foldable polygon arcs, to more complicated foldable arc-modules, to foldable perfect circle modules (a zero or equilibrium element), to supercircle modules, to super-duper "circles?" How could we define a metric for this progression?*

"There is no branch of mathematics, however abstract, which may not someday be applied to phenomena of the real world."

— Nikolai Ivanovich Lobatchevsky

Applications

- *Objects of art*
- *Spherical trigonometry education (a lost subject!)*
- *Math education in general*
- *Polyhedra Geometry*
- *Graph theory*
- *Combinatorics*
- *Group Theory*
- *Geodesics (Fuller did this work before developing the dome)*
- *Electron orbits*
- *Cell growth*
- *Protein folding*
- *Origami*
- *Other paper folding model-building techniques (e.g., Prof. Hilton wrote a book with Jean Pederson "Build Your Own Polyhedra" which discusses folding long strips of paper)*

The Supercircle Poem

By Jeannie Moberly

Two circles isometrical lived in sweet harmony
Began to brawl, could not agree on whose was whose degree
The poor sad cone was left alone quite unequivocal
And on a lark with extra arc ran selfish supercircle.

References

[1] Buckminster Fuller, Synergetics, Macmillan, Vol. 1, pp. 164–189, 1975. The on-line account is at http://www.rwgrayprojects.com/synergetics/s04/p5000.html.

[2] Kaj Nielsen and John Vanlonkhuyzen, Plane and Spherical Trigonometry, Barnes & Noble, pp. 103–151, 1946.

[3] Robert W. Gray, Notes to R. B. Fuller's Synergetics: Appendices, 1993. Additional notes at http://www.rgrayprojects.com/rbfnotes/greatc/greatc1.html.

[4] Edward Suzuki Hoerdt, Atommetrics: Another View of Atomic Structure Based on Electron Orbital Geometry, Forma, Vol 17, No 4, pp. 275–350, 2002.

[5] H. S. M. Coxeter, Introduction to Geometry, John Wiley & Sons, Inc., second edition, pp. 270–277.

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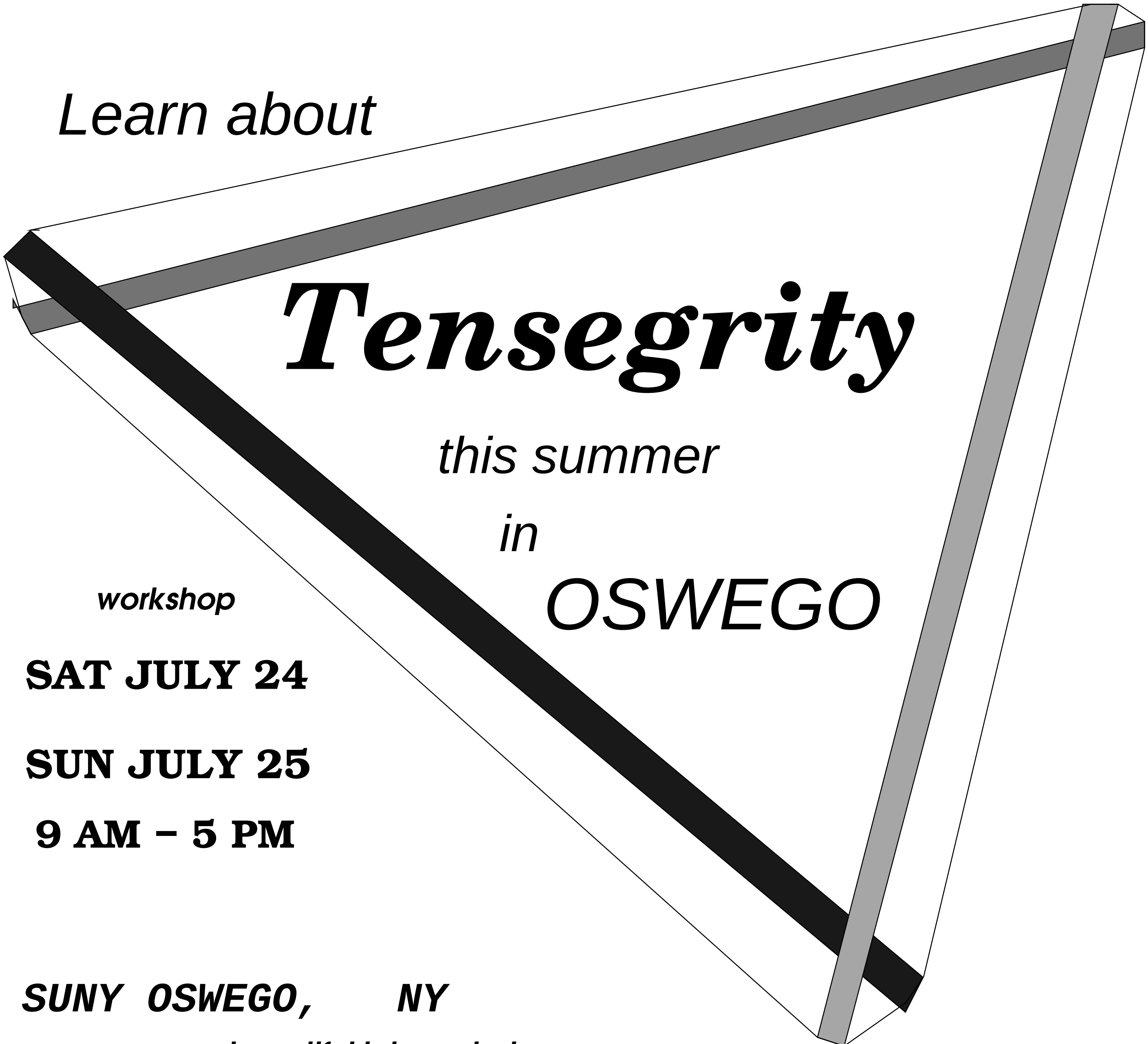

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JULY 5 – JULY 23 : SYNERGETICS IN THE CLASSROOM

taught by Joe Clinton, facilitated by John Belt

to participate in the course contact John Belt <belt@oswego.edu>

SNEC: Synergeticists Of The NorthEast Corridor http://SNEC.Synergeticists.org

Synergetics is the discipline of holistic thinking which R. Buckminster Fuller introduced and began to formulate. Synergetics provides a method and a philosophy for problem-solving and design and therefore has applications in all areas of human endeavor. Synergetics is multi-faceted: it involves geometric modeling, exploring inter-relationships in the facts of experience and the process of thinking. Synergetics endeavors to identify and understand the methods that Nature actually uses in coordinating Universe (both physically and metaphysically).

Synergetics is studied by a wide-range of people from artists to business people to teachers and truck drivers to scientists, architects, designers, mathematicians and engineers. SNEC is an organization to bring together these diverse people in face-to-face workshops, symposia, seminars, and other ad-hoc or planned meetings to better understand the many facets of this diverse discipline and its methods.

Report on the March 2004 SNEC Symposium:
http://SNEC.Synergeticists.org/snec.meeting.2004.03.html

Report on the June 2003 SNEC Workshop:
http://SNEC.Synergeticists.org/snec.meeting.2003.06.html